# Solution to Homework 3 (ECE3075 Summer 2004)

1. (10%) Work out Problem 3-1.2 in Cooper and McGillem.
   Two r. v.'s $X$ and $Y$ have a joint pdf of the form: $f(x,y) = kxy \;\; 0 \le x,\; y \le 1$
   a) To make $f(x, y)$ a valid pdf: $\int_0^1 \int_0^1 f(x,y)dydx = k/4 = 1 \Rightarrow \; k = 4$
   b) $F(x,y) = \int_0^x \int_0^y 4xydydx = x^2 y^2, \;\; 0 \le x, y \le 1$
   c) $P(0 \le X \le 1/2, 1/2 \le Y \le 1) = \int_0^{1/2} \int_{1/2}^1 4xydydx = \frac{1}{4} * \frac{3}{4} = \frac{3}{16}$
   d) $f_X(x) = \int_0^1 4xydy = 2x, \;\; 0 \le x \le 1$

2. (10%) Work out Problem 3-2.1 in Cooper and McGillem.
   $X$ is a Rayleigh r. v. with mean 10, and $N$ is a uniform r. v. with mean 0 and variance 12, $X$ and $N$ are independent and only observed through $Y$=$X$+$N$. First, $f_X(x) = \frac{x}{\sigma^2}\exp[-\frac{x^2}{2\sigma^2}], \; x \ge 0$. Since E[X]=10=$\sqrt{\pi/2}\,\sigma$, we have $\sigma = 10\sqrt{2/\pi} = 7.979 \approx 8$.
   Next if $N$ has a range of [-b,b] (the only way a uniform r. v. can be zero mean), then $E(N^2) = [b-(-b)]^2 / 2b = 2b = 12 \Rightarrow b = 6,$ and $f_N(n) = 1/12, \; -6 \le n \le 6$
   a) Find conditional pdf, $f(x|y)$ for $y$=0, 6, and 12; Using Bayes' theorem,

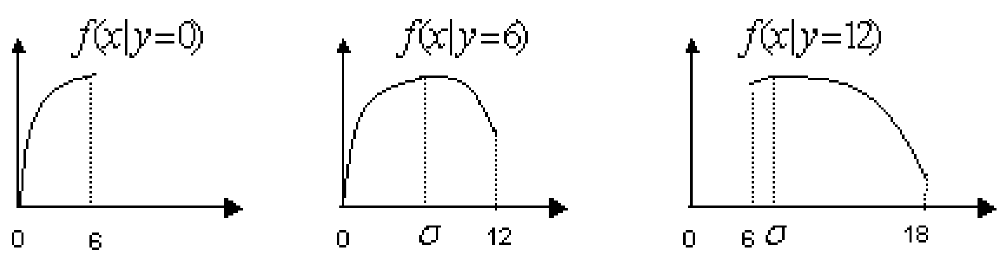
$$f_X(x|y) = \frac{f_N(y-x) f_X(x)}{f_Y(y)} = \frac{1}{12 f_Y(y)} \frac{x}{\sigma^2}\exp[-\frac{x^2}{2\sigma^2}], \; y-6 \le x \le y+6, \; y \ge -6$$

   now $\int_{y-6}^{y+6} \frac{x}{12\sigma^2}\exp[-\frac{x^2}{2\sigma^2}]dx$, with $x \ge 0$ to have a valid integral, so we have

$$f_Y(y) = \int_0^{y+6} \frac{x}{12\sigma^2}\exp[-\frac{x^2}{2\sigma^2}]dx = \frac{1}{12}[1 - e^{-\frac{(y+6)^2}{2\sigma^2}}], \; \text{if } -6 \le y \le 6$$

$$f_Y(y) = \int_{y-6}^{y+6} \frac{x}{12\sigma^2}\exp[-\frac{x^2}{2\sigma^2}]dx = \frac{1}{12}[e^{-\frac{(y-6)^2}{2\sigma^2}} - e^{-\frac{(y+6)^2}{2\sigma^2}}], \; \text{if } y > 6;$$ so for

   $y$=0, 6 and 12, we have the following conditional pdf's, $p(x|y)$, respectively:




   b) For $y=12$, maximum $f(x|y)$ occurs at $\hat{x} = \arg\max_x f(x \mid y = 12) = \sigma$.

3. (10%) Work out Problem 3-3.3 in Cooper and Mcgillem.
   If X and Y and independent and Gaussian with means 1 and 2, and variances 1 and 4, respectively, find the probability that $XY > 0$:

$$f(x,y)=\{\frac{1}{\sqrt{2\pi}}\exp[-\frac{(x-1)^2}{2}]\}*\{\frac{1}{\sqrt{2\pi}*2}\exp[-\frac{(y-2)^2}{2*2^2}]\}=f_X(x)f_Y(y)$$

So $P(XY>0)=P(X>0,Y>0)+P(X<0,Y<0)$

$$=\int_0^\infty\int_0^\infty f(x,y)dydx+\int_{-\infty}^0\int_{-\infty}^0 f(x,y)dydx$$

$$=[\int_0^\infty f_X(x)dx]*[\int_0^\infty f_Y(y)dy]+[\int_{-\infty}^0 f_X(x)dx]*[\int_{-\infty}^0 f_Y(y)dy]$$

With the definition of *Q(x)* in Eq. (2-20) and the property in Eq. (2-21), we have

$$P(XY>0)=Q(\frac{0-1}{1})*Q(\frac{0-2}{2})+[1-Q(\frac{0-1}{1})]*[1-Q(\frac{0-2}{2})]$$

$$=Q(-1)*Q(-1)+[1-Q(-1)]*[1-Q(-1)]$$

$$=Q(-1)*Q(-1)+Q(1)*Q(1)=0.7330$$

4. (10%) Work out Problem 3-4.3 in Cooper and McGillem.

Assume *X* and *Y* are independent of *X* with variance of 9 and 25, respectively. Define Z=X+Y,

a) $$\rho_{XY}=E[\frac{X-\mu_X}{\sigma_X}*\frac{Y-\mu_Y}{\sigma_Y}]=\frac{1}{\sigma_X\sigma_Y}[E(X-\mu_X)E(Y-\mu_Y)]$$

$$=\frac{1}{\sigma_X\sigma_Y}[E(X)E(Y)-\mu_X\mu_Y]=0$$ (Remember X and Y ae independent);

b) $$\rho_{YZ}=E[\frac{Y-\mu_Y}{\sigma_Y}*\frac{(X+Y-\mu_X-\mu_Y)}{\sigma_Z}]=\frac{1}{\sigma_Y\sigma_Z}[E(X-\mu_X)E(Y-\mu_Y)+E[(Y-\mu_Y)^2]]$$

$$=\frac{1}{\sigma_Y\sqrt{(\sigma_X^2+\sigma_Y^2)}}[0+\sigma_Y^2]=5/\sqrt{34}=0.857$$ (Need the variance computed in Part (c));

c) $$\sigma_Z^2=E[(X+Y-\mu_X-\mu_Y)^2]=E[(X-\mu_X)^2]+2E[(X-\mu_X)(Y-\mu_Y)]+E[(Y-\mu_Y)^2]$$

$$=\sigma_X^2+\sigma_Y^2=9+25=34.$$

5. (10%) Work out Problem 3-5.2 in Cooper and Mcgillem.

*X* is uniform between 8:00 and 8:10, and *Y* is uniform between 7:55 and 8:05. *X* and *Y* are independent, and *Z*=*X*-*Y* is the r. v.

a) $$f_Z(z)=\int_{-\infty}^\infty f_Y(y)f_X(z+y)dy=\int_{8:00-z}^{8:05}\frac{1}{10}*\frac{1}{10}dy=\frac{1}{100}(5+z),\quad -5\le z\le 5$$

$$=\int_{7:55}^{8:10-z}\frac{1}{10}*\frac{1}{10}dy=\frac{1}{100}(15-z),\quad 5\le z\le 15$$

b) $P(\text{"catching the train"})=P(Z>0)=1-\int_{-5}^0\frac{1}{100}(5+z)dz=1-\frac{25}{200}=\frac{7}{8}=0.875;$

c) $P(\text{"catching the train with 3 min delay"})=P(Z>3)=1-\int_{-5}^3\frac{1}{100}(5+z)dz=1-\frac{64}{200}=0.68.$

6. (20%) Work out Problem 3-5.7 in Cooper and Mcgillem (MATLAB exercise).

7. (10%) Work out Problem 3-6.1 in Cooper and Mcgillem.

   $f(x,y)=4xy,\ \ 0<x<1, 0<y<1,$ and $Z=X+Y$. Let $W=X$, and compute the inverse functions, we have $X=W$, $0<W<1$, and $Y=Z-W$, $0<W<Z$ for $0<Z<1$ and $Z-1<W<1$ for $1<Z<2$. We also have the Jacobian J=-1. Therefore the pdf of $Z$ is

$$g(z)=\int_{-\infty}^{\infty} 4w(z-w)dw = \int_0^z (2zw^2-4w^3/3)dw = 2z^3/3,\ \ 0<z<1,$$

$$=\int_{z-1}^{1}(2zw^2-4w^3/3)dw = 2[-z^3+6z-4]/3,\ \ 1<z<2.$$

8. (10%) Work out Problem 3-7.2 in Cooper and Mcgillem.

   a) A Gaussian pdf is $f(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp[-\frac{x^2}{2\sigma^2}]$, find the characteristic function

$$\phi(u)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp[-\frac{x^2}{2\sigma^2}]\exp[jux]dx$$

$$=\exp[-\frac{u^2\sigma^2}{2}]*\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp[-(\frac{x}{\sqrt{2}\sigma}-\frac{ju\sigma}{\sqrt{2}})^2]dx=\exp[-\frac{u^2\sigma^2}{2}]$$

$$E[X^n]=\frac{1}{j^n}[\frac{d^n\phi(u)}{du^n}]|_{u=0}$$

$$E[X]=\frac{1}{j}[\frac{d\phi(u)}{du}]|_{u=0}=-u\sigma^2\exp[-\frac{u^2\sigma^2}{2}]|_{u=0}=0$$

   b) $E[X^2]=\frac{1}{j^2}[\frac{d^2\phi(u)}{du^2}]|_{u=0}=\{\sigma^2\exp[-\frac{u^2\sigma^2}{2}]-u^2\sigma^4\exp[-\frac{u^2\sigma^2}{2}]\}|_{u=0}=\sigma^2$

$$E[X^4]=[\frac{d^4\phi(u)}{du^4}]|_{u=0}=\{3\sigma^4\exp[-\frac{u^2\sigma^2}{2}]-3u^2\sigma^6\exp[-\frac{u^2\sigma^2}{2}]+u^4\sigma^8\exp[-\frac{u^2\sigma^2}{2}]\}|_{u=0}=3\sigma^4$$

   $\cdots\cdots E[X^n]=1*3*\cdots*(n-1)\sigma^n$ for $n$ even and =0 for $n$ odd (proof by induction).

9. (10%) Work out Problem 3-7.3 in Cooper and Mcgillem.

   $\phi(u)=1-p+pe^{ju}$ the characteristic function of a Bernoulli r. v.

   a) $E[X]=\frac{1}{j}\frac{d\phi(u)}{du}|_{u=0}=pe^{ju}|_{u=0}=p$

   b) $E[X^2]=\frac{1}{j^2}\frac{d^2\phi(u)}{du^2}|_{u=0}=pe^{ju}|_{u=0}=p$, so $E[(X-\bar{X})^2]=E[X^2]-\bar{X}^2=p(1-p)$

$$E[X^3]=\frac{1}{j^3}\frac{d^3\phi(u)}{du^3}|_{u=0}=pe^{ju}|_{u=0}=p,$$

   c) so $E[(X-\bar{X})^3]=E[X^3]-3E[X^2]\bar{X}+3E[X]\bar{X}^2-\bar{X}^3$

$$=p-3p*p+3p*p^2-3p^3=p-3p^2.$$