Solution to Homework 2 (ECE3075 Summer 2004)

1. (10%) Work out Problem 2-1.2 in Cooper and Mcgillem.
   State each of the following r. v. is continuous or discrete, and its outcome:
   a) Rolling a pair of dice: discrete with outcomes 2 - 12
   b) Measuring the voltage of a 12-V battery: continuous, 11.5 – 12.5 V
   c) Randomly selecting a telephone number: discrete, 000-0000 to 999-9999
   d) Weighing adult males: continuous, 100 – 500 pounds

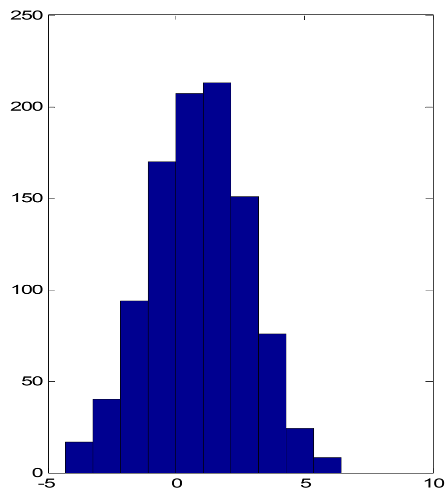
2. (10%) Work out Problem 2-2.3 in Cooper and Mcgillem.
   A PDF has the form: $F_X(x) = A\{1-\exp[-(x-1)]\},\ 1 \le x < \infty,\ F_X(x) = 0,\ -\infty < x < 1$,
   a) Since $F_X(\infty) = A$, we need $F_X(\infty) = 1$ or $A=1$ to make it a valid PDF;
   b) $F_X(2) = \{1-\exp[-(2-1)]\} = 1-1/e = 0.6321$;
   c) $P(2 < X < \infty) = F_X(\infty) - F_X(2) = 1-0.6321 = 0.3679$
   d) $P(1 < X \le 3) = F_X(3) - F_X(1) = 0.8647$

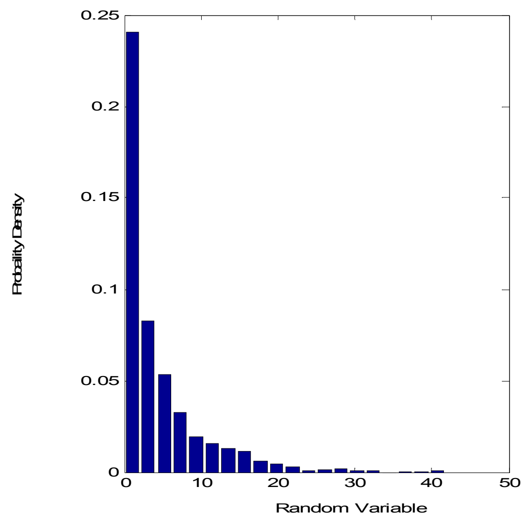
3. (10%) Work out Problem 2-3.3 in Cooper and Mcgillem.
   Assume a pdf of the form: $f_X(x) = \exp(-2|x|),\ -\infty < x < \infty,$:
   a) If $Y = X^2$ then $f_Y(y) = \frac{1}{2\sqrt{y}}[f_X(\sqrt{y}) + f_X(-\sqrt{y})] = \frac{1}{\sqrt{y}}[\exp(-2\sqrt{y})],\ y \ge 0$
   b) $$P(2 < Y) = \int_2^\infty f_Y(y)dy = \int_2^\infty \frac{1}{\sqrt{y}}\exp[-2\sqrt{y}]dy \quad (\text{let } z^2 = y, \text{ and } dy = 2zdz)$$ $$= \int_{\sqrt{2}}^\infty \frac{1}{z}\exp[-2z]*2zdz = e^{-2\sqrt{2}} = 0.0591$$

4. (10%) Work out Problem 2-4.3 in Cooper and Mcgillem.
   Assume a pdf of the form: $f_Y(y) = Ky,\ 0<y \le 6,\ f_Y(y) = 0,$ otherwise:
   a) $\int_0^6 Kydy = 1$ or $K6^2/2 = 1 \Rightarrow K = \frac{1}{18}$
   b) $\overline{Y} = \int_0^6 y*\frac{1}{18}ydy = 6^3/54 = 4$
   c) $\overline{Y}^2 = \int_0^6 y^2*\frac{1}{18}ydy = 6^4/72 = 18$
   d) $\sigma^2 = \overline{Y}^2 - (\overline{Y})^2 = 18 - 4^2 = 2$
   e) $E[(Y-\overline{Y})^3] = \int_0^6 (y-4)^3*\frac{1}{18}ydy = \frac{6^5}{90} - \frac{12*6^4}{72} + \frac{48*6^3}{54} - \frac{64*6^2}{36} = -1.6$
   f) $E[Y^n] = \int_0^6 y^n*\frac{1}{18}ydy = 6^{n+2}/[18*(n+2)] = 2*6^n/(n+2)$

5. (10%) Work out Problem 2-5.1 in Cooper and Mcgillem.
   A Gaussain random voltage V has a mean of 5 and a variance of 16.
   a) $P(V > 0) = \int_0^\infty \frac{1}{\sqrt{2\pi*16}}\exp[-\frac{1}{2*16}(v-5)^2]dv = Q(\frac{0-5}{4}) = 0.8944$
   b) $P(0 < V \le 5) = \int_0^5 \frac{1}{\sqrt{2\pi*16}}\exp[-\frac{1}{2*16}(v-5)^2]dv = Q(\frac{0-5}{4}) - Q(0) = 0.3944$
   c) $P(V > 10) = \int_{10}^\infty \frac{1}{\sqrt{2\pi*16}}\exp[-\frac{1}{2*16}(v-5)^2]dv = Q(\frac{10-5}{4}) = 0.1057$

6. (10%) Work out Problem 2-5.6 in Cooper and Mcgillem (MATLAB exercise).





7. (10%) Work out Problem 2-6.6 in Cooper and Mcgillem.
   a) and b) X^2 is $\chi^2(5)$, so mean is 5 and variance is 10
   c) To find the maximum of the density function, we differentiate $f_{\chi^2}(u)$ and set it to zero, we have $(\frac{n}{2}-1)*\hat{u}^{-1}=\frac{1}{2}$ or equivalently $\hat{u}=n-2=3$

8. (10%) Work out Problem 2-7.4 in Cooper and Mcgillem.
   The lifetime of a light bulb is modeled by an Erlang r. v. with $\tau = 2000$ and $k = 4$ with the PDF defined as: $F_k(t)=\int_0^t f_k(w)dw=\int_0^t \frac{w^{k-1}}{\tau^k (k-1)!}\exp(-w/\tau)dw$
   a) $E(T)=k\tau=4*2000=8000$
   b) $P(T\geq 10000)=1-F_4(10000)$, using $\int x^3 e^{ax}dx=\frac{e^{ax}}{a^4}(a^3x^3-3a^2x^2+6ax-6)$, we have
   $F_4(t)=1-\frac{1}{6}e^{-\frac{t}{2000}}[(\frac{t}{2000})^3+3(\frac{t}{2000})^2+6(\frac{t}{2000})+6)]$ or $P(T\geq 10000)=\frac{e^{-5}}{6}(125+75+30+6)=0.265$
   c) $P(T<4000)=F_4(4000)=1-\frac{e^{-2}}{6}(2^3+3*2^2+6*2+6)=0.14288$

9. (10%) Work out Problem 2-8.3 in Cooper and Mcgillem.
   The distance from the target center is a Rayleigh random variable with a pdf $f_R(r)=(r/\sigma^2)\exp[-r^2/2\sigma^2],\ r\geq 0,$ or $F_R(r)=P(R\leq r)=1-\exp[-r^2/2\sigma^2],\ r\geq 0,$ since $F_R(0.5)=1-1/10,$ we have $F_R(0.5)=1-\exp[-1/2\sigma^2]=0.9 \Rightarrow \sigma=0.2330$
   a) Let $A=\{R\leq 0.1\}$ and $M=\{R\leq 0.5\}$ with $F_R(0.5)=1-0.1=0.9,$ then
   $P(A\,|\,M)=P(A,M)/P(M)=P(A)/P(M)=F_R(0.1)/F_R(0.5)=0.0978$
   b) Let $B=\{0.5\leq R\leq 0.8\}$ and $Q=\{R>0.5\}$, then
   $P(B\,|\,Q)=P(B)/P(Q)=[F_R(0.8)-F_R(0.5)]/[1-F_R(0.5)]=0.9725$

10. (10%) Work out Problem 2-9.3 in Cooper and Mcgillem.

A limiter has an I-O characteristic defined by: $V_o = \begin{cases} -B & V_i < -A \\ BV_i / A & -A \le V_i \le A \\ B & V_i > A \end{cases}$

a) If the input is a Gaussian r. v. with mean $\bar{V}_i$ and variance σ_i^2, then the output is a clipped Gaussian r. v. with mean $\bar{V}_o = B\bar{V}_i / A$ and variance $\sigma_o^2 = B^2\sigma_i^2 / A^2$, such that $f(V_o) = \begin{cases} Q & V_o = -B \\ \frac{1}{\sqrt{2\pi\sigma_o^2}} \exp[-\frac{1}{2\sigma_o^2}(V_o - \bar{V}_o)^2] & -B < V_o < B \\ Q & V_o = B \end{cases}$. To make it a valid density, we need to have a discontinuity at $V_o = \pm B$, and $f(V_o) = 0,$ otherwise.. We also need $Q = \int_B^\infty f(V_o) dV_o$ to be a probability mass at $V_o = \pm B$

b) If V_i is uniform from -2 to 8, with A=B=5, then $f(V_i) = 1/10,\ -2 \le V_i \le 8$, and $f(V_o) = 1/10,\ -2 < V_o < 5,\ f(V_o) = 3/10,\ V_o = 5,$ and $f(V_o) = 0,$ otherwise, so the mean is $E(V_o) = \int_{-2}^{5} V_o / 10 dV_o + 5 * \frac{3}{10} = \frac{51}{20} = 2.55$