ECE3075 - Random Signals

Chapter 5: Random (Stochastic) Processes

Chin-Hui Lee
School of Electrical and Computer Engineering
Georgia Institute of Technology
Atlanta, GA 30332, USA
chl@ece.gatech.edu

Introduction to Random Process

- Random Process: a collection of time function, and an associated probability description
  - Ensemble: $\{x(t)\}$, with $x(t)$ a sample function
  - An arbitrary random sample function, denoted as $X(t)$
  - At any time instance t_1, a random variable $X(t_1) = X_1$
- Extension from random variables to random processes
  1. Continuous vs. discrete
  2. Deterministic vs. non-deterministic
  3. Stationary vs. non-stationary
  4. Ergodic vs. non-ergodic
  5. Quasi-stationary random processes: e.g. speech signal
- Many engineering and other applications

Continuous vs. Discrete Processes

- Defined by the random variable at time *t*, *X(t)*
  - Continuous Process: *X(t)* is a continuous r. v. (Fig. 5-1)
  - Discrete Process: *X(t)* is a discrete r. v. (Fig. 5-2)
  - Mixed Process: *X(t)* has a mixed definition (Fig. 5-3)
- Point processes or Time Series
  - Many economic indicators, e.g daily DJ Index close
  - Signals and Systems: more later (ECE2025: DSP)
  - Sampling in time: Analog to digital conversion
  - Many natural phenomena, e.g. sun spot
- From Time Series to Spatial Process
- Illustrations: Exercises 5-2.1 and 5-2.2

Deterministic vs. Non-deterministic Processes

- Defined by the random variable at time *t*, *X(t)*
  - Non-deterministic: *X(t)* is random
  - Deterministic: *X(t)* is of a fixed value at time *t*
- Three examples of deterministic processes

1. $X(t) = A\cos(\omega t + \theta)$, A and ω are constant and θ is random

2. $X(t) = \sum_{n=0}^{\infty} [A_n \cos(2\pi f_0 t) + B_n \cos(2\pi f_0 t)]$,

   A_n and B_n are random, but can be determined by past history

3. $X(t) = A\exp(-\beta t)$, A and β are random, but fixed once sampled

- Illustrations: Exercises 5-3.1 and 5-3.2

Stationary vs. Non-stationary Processes

- Defined by the random variable at time *t*, $X(t)$
  - Stationary Process: all marginal and joint distributions of $X(t)$ are independent of time index t
  - Mathematical convenience and quasi-stationary processes
- Back to deterministic process examples
  - In Ex. 1, $X(t)$ is stationary only if θ is uniform from 0 to 2π
  - In Ex. 2, $X(t)$ is stationary only if A_n and B_n are zero mean, i.i.d. Gaussian for the same index n
  - In Ex. 3, $X(t)$ is non-stationary in all circumstances
- Stationarity in a wide sense: a relaxed requirement
  - mean value of $X(t)$ is the same for all t
  - $E[X(t)X(s)]$ depends only on the time difference t-s (later)
- Illustrations: Exercises 5-4.1 and 5-4.2

Ergodic vs. Non-ergodic Processes

- Defined by the random sample function, *X(t)*
  - Ergodicity: the ensemble exhibits same statistical behaviors that it is possible to determine the ensemble behaviors by sampling one typical sample function
  - An ergodic process has to be also a stationary process

$$E[X^n(t)] = \int_{-\infty}^{\infty} x^n(t) f(x) dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X^n(t) dt$$

- An examples of stationary but non-ergodic processes
  - $X(t) = A\cos(\omega t + \theta)$, ω is a constant, and A and θ are independent random variables, A is a r. v. respect to the emsemble and θ is uniform over 0 and 2π
- Illustrations: Exercises 5-5.1 and 5-5.2

Measurement of Process Parameters

- Some limitations:
  - sampling in ensemble: few sometime only one
  - sampling is time: time span is usually finite
  - knowing the category of the process of interest
- Mean value of an ergodic process: r. v. $\hat{\bar{X}} = \frac{1}{T}\int_0^T X(t)dt$

$$E[\hat{\bar{X}}] = \frac{1}{T}\int_0^T E[X(t)]dt = \frac{1}{T}\int_0^T \bar{X}dt = \bar{X}$$

$$E[\hat{\bar{X}}] = E[\frac{1}{N}\sum_{i=1}^{N} X(t_i)] = \frac{1}{N}\sum_{i=1}^{N} E[X_i] = \bar{X}$$

$$E[X_i X_j] = E[X^2]\ (i=j),\ E[X_i X_j] = \bar{X}^2\ \ (i \neq j,\ \text{independent})$$

$$\text{Var}[\hat{\bar{X}}] = E[\hat{\bar{X}}^2] - (E[\hat{\bar{X}}])^2 = \sigma_X^2 / N$$

Center of Signal and Image Processing
Georgia Institute of Technology
CSIP

An Example: Data Smoothing with a Moving Average Window

- Moving Average over a finite-size time window

$$\hat{X}_i = \frac{1}{n_L + n_R + 1} \sum_{k=-n_L}^{n_R} Y_{i+k} \quad \text{(with } Y_j = X_j + N_j \text{ over a time window)}$$

- Smoothing observed data: a practical problem
  - needed for many noisy samples of engineering data
  - the size of the window depends on the problem at hand
  - many techniques in Chapter 3 can be applied
  - smoothing will also smear the variance
- Illustration Example: Figure 5-4

Summary

- **Today's Class**
  – Random (or Stochastic) Processes
- **Reading Assignments**
  – Cooper & McGillem, Chapter 5
- **Class Next Week**
  – Chapter 6