## SOLUTIONS to Final Exam

**Problem 1 (20%)**

Let $X$ and $Y$ be two independent standardized (zero mean, unit variance) Gaussian random variables. Define another two additional random variables $U=aX+bY$, and $V=aX-bY$, with $a$ and $b$ being two positive scaling constants.

(a) Find the means and variances of $U$ and $V$;

(b) Find the correlation $E[UV]$ and correlation coefficient of $U$ and $V$.

(c) Use the Jacobian transformation to find the joint probability density function of $U$ and $V$, expressing it in the bivariate Gaussian density form [Recall that the Jocobian matrix $J$ is defined as a 2x2 matrix with $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, $\partial y/\partial v$ as its elements];

(d) Make sure your answers in Parts (a) and (b) agree with the pdf in Part (c);

(e) Find the marginal probability density functions, $f_U(u)$ and $f_V(v)$. Are the marginal densities depending on the value of the correlation coefficient?

(f) Under what conditions can $U$ and $V$ be independent? State your reasons.

**Solution:**

(a) $E[U] = E[aX] + E[bY] = 0$ and $E[V] = E[aX] - E[bY] = 0$,

$\sigma_U^2 = \mathrm{Var}[aX] + \mathrm{Var}[bY] = a^2 + b^2$, $\sigma_V^2 = \mathrm{Var}[aX] + \mathrm{Var}[bY] = a^2 + b^2$;

(b) $E[UV] = E[a^2X^2 - b^2Y^2] = a^2 - b^2$, $\rho = E[UV]/[\sigma_U\sigma_V] = (a^2 - b^2)/(a^2 + b^2)$;

(c) First perform the mapping, we have $x = \frac{1}{2a}(u+v)$ and $y = \frac{1}{2b}(u-v)$, so we have

$\partial x/\partial u = \frac{1}{2a}, \partial x/\partial v = \frac{1}{2a}, \partial y/\partial u = \frac{1}{2b}, \partial y/\partial v = -\frac{1}{2b}$, and $|\det(J)| = |-\frac{1}{2ab}| = \frac{1}{2ab}$

the joint pdf $f(x,y) = \frac{1}{2\pi}\exp[-\frac{1}{2}(x^2+y^2)] = f_X(x) * f_Y(y)$, so we have

$$g(u,v) = |J|\, f(\Psi_1(u,v), \Psi_2(u,v)) = \frac{1}{2ab*2\pi}\exp[-\frac{1}{2}\{\frac{(u+v)^2}{4a^2} + \frac{(u-v)^2}{4b^2}\}]$$

$$= \frac{1}{2ab*2\pi}\exp[-\frac{1}{2}\{(\frac{1}{4a^2}+\frac{1}{4b^2})(u^2+v^2) - (-\frac{1}{4a^2}+\frac{1}{4b^2})2uv\}]$$

$$= \frac{1}{\sqrt{(2\pi)^2*(1-\rho^2)\sigma_U^2\sigma_V^2}}\exp[-\frac{1}{2(1-\rho^2)}\{\frac{u^2}{\sigma_U^2}+\frac{v^2}{\sigma_V^2}-\frac{2\rho uv}{\sigma_U\sigma_U}\}]$$

$$\Rightarrow \rho = \frac{(-1/4a^2+1/4b^2)}{(1/4a^2+1/4b^2)} = \frac{(a^2-b^2)}{(a^2+b^2)}$$ (from the above bivariate Gaussian pdf);

(d) The answers in Parts (a) and (b) agree with the parameters in the pdf in Part (c);

(e) Since $1-\rho^2 = 4a^2b^2/(a^2+b^2)^2$, and $\sigma_U^2 = \sigma_V^2 = \sigma_U\sigma_V = a^2+b^2$, we have the exponent in $g(u,v)$ can be simplied to form a perfect square, i.e.

$$\frac{1}{2(1-\rho^2)}(\frac{u^2}{\sigma_U^2}+\frac{v^2}{\sigma_V^2}-\frac{2\rho uv}{\sigma_U\sigma_U}) = \frac{(a^2+b^2)}{2*(4a^2b^2)}*[(u-\rho v)^2] + \frac{1}{2*(a^2+b^2)}v^2$$

$$f_V(v) = \int_{-\infty}^{\infty} g(u,v)du = \frac{1}{\sqrt{2\pi\sigma_{U|V}^2}}\int_{-\infty}^{\infty}\exp[-\frac{(u-\rho v)^2}{2\sigma_{U|V}^2}]du * \frac{1}{\sqrt{2\pi\sigma_V^2}}\exp[-\frac{v^2}{2\sigma_V^2}]$$

$$\Rightarrow f_V(v) = \frac{1}{\sqrt{2\pi\sigma_V^2}}\exp[-\frac{u^2}{2\sigma_V^2}]$$ [Note: $\sigma_{U|V}^2 = \frac{4a^2b^2}{(a^2+b^2)} = (1-\rho^2)\sigma_U^2$].

Similarly $f_U(u) = \frac{1}{\sqrt{2\pi\sigma_U^2}}\exp[-\frac{v^2}{2\sigma_U^2}] \Rightarrow f_U(u)$ and $f_V(v)$ do not depend on $\rho$;

(f) $U$ and $V$ are independent Gaussain r. v.'s if $E[UV] = 0$, or $a = b$;

**Problem 2 (20%)**

A passenger can take two different bus lines to work. Both lines stop at the same bus stop. Let $X$ and $Y$ be independent random variables representing the times of arrival for the bus of line 1 and line 2, respectively. It is known that $X$ and $Y$ have exponential densities with means of 5 and 10 minutes, respectively. Let $Z$ be the random variable representing time the passenger will wait for a bus, i.e. $Z$=min($X$, $Y$). [Hint: $f_X(x) = K_x e^{-a_x x} u(x)$, and $f_Y(y) = K_y e^{-a_y y} u(y)$, with *u(.)* a step function]

(a) Find the pairs of values of $(K_x, a_x)$ and $(K_y, a_y)$ to make $f_X(x)$ and $f_Y(y)$ valid densities, and with the corresponding means of 5 and 10 minutes ;

(b) Find *P(Z>5)*, the probability that after 5 minutes no bus has arrived yet. Interpret your result and make sure the answer is reasonable [Hint: divide the *(x,y)* plane into two regions : $R_1 = \{Y \ge X\}$ and $R_2 = \{X > Y\}$];

(c) Find the probability density function of $Z$ [Hint: it is obtained by taking the derivative of the corresponding distribution function, $F_Z(z) = P(Z \le z)$];

(d) What is the average time, *E[Z]*, the passenger will wait at the bus stop?

(e) Compare *E[Z]* with *E[X]* and *E[Y]*, and make sure your answer in Part (d) is intuitively reasonable. Check this answer with the result in Part (b).

**Solution:**

(a) $\int_0^\infty K_x e^{-a_x x} dx = -\frac{K_x}{a_x} e^{-a_x x} |_0^\infty = \frac{K_x}{a_x} = 1 \Rightarrow K_x = a_x$, since $E[X] = 5$,

we have $E[X] = \int_0^\infty a_x x e^{-a_x x} dx = (-a_x x e^{-a_x x} - \frac{1}{a_x} e^{-a_x x}) |_0^\infty = \frac{1}{a_x} \Rightarrow a_x = \frac{1}{5}, K_x = \frac{1}{5}$;

similarly since $E[Y] = 10 \Rightarrow a_y = \frac{1}{10}, K_y = \frac{1}{10}$;

(b) Partition the $(x, y)$ plane into two regions, $R_1 = \{Y \ge X\}(Z = X)$ and $R_2 = \{Y < X\}(Z = Y)$

For any given $z \ge 0$, $P(Z > z) = P(X > z, R_1) + P(Y > z, R_2)$

$$= \int_z^\infty [\int_x^\infty f_Y(y)dy] f_X(x)dx + \int_z^\infty [\int_y^\infty f_X(x)dx] f_Y(y)dy$$

$$= \int_z^\infty [(-e^{-y/10}) |_x^\infty] \frac{1}{5} e^{-x/5} dx + \int_z^\infty [(-e^{-x/5}) |_y^\infty] \frac{1}{10} e^{-y/10} dy$$

$$= e^{-\frac{3}{10}z}, \text{ for } z = 5, P(Z > 5) = e^{-\frac{3}{2}} = 0.2231;$$

(c) Since $F_Z(z) = P(Z \le z) = 1 - P(Z > z) \Rightarrow f_Z(z) = dF_Z(z)/dz = \frac{3}{10} e^{-\frac{3}{10}z}$;

(d) $E[Z]$=10/3, the expected waiting time;

(e) Clearly $E[Z]$ is expected to be less than either $E[X] = 5$ or $E[Y] = 10$,

because the passager always takes the minimum waiting time with $Z$=min$(X, Y)$.

**Problem 3 (20%)**

You play with a friend the following game: you flip a coin provided by your friend and if the result is Head you win $1 while if it is a Tail you lost $1 (i.e. you win -\$1). Let $X_i$ be the random variable describing the amount you win at the $i$-th coin flip. You play the game $n$=2,500 times. Let $V$ be the number of winning games, and $S = \sum_{i=1}^{n} X_i$ be the random variable representing the total winning amount after these plays. First assume a fair coin is used, i.e. $p=P(X=1)= P(X=-1)=0.5$.

(a) $V$ is known to have a Binomial distribution with $P_n(V = v) = {}_nC_v p^v (1-p)^{n-v}$, where ${}_nC_v = \frac{n!}{v!(n-v)!}$, show that $E[V]=np$, and Var[$V$]=$np(1-p)$;

(b) According to the Law of Large Number, $V$ is approximately normal when $n$ is large, find the approximate probability density functions of $V$ and $S$;

(c) What is the probability that at the end of the game you lost more than \$300, i.e. find $P(S<-300)$ [Hint: an approximation is fine, no need to be exact];

(d) At the end of the game you lost \$300. You suspect your friend cheated and the coin is biased. Estimate the value of the success rate, $p=P(X=1)$;

(e) Construct a 99% confidence interval for the unknown parameter, $p$;

(f) Do you think the coin is fair? Why?

**Solution:**

(a) $P_n(V = v) = {}_nC_v p^v (1-p)^{n-v}$

$$E[V] = \sum_{v=0}^{n} \frac{n!}{v!(n-v)!} v p^v (1-p)^{n-v}$$

$$= np * \sum_{v=1}^{n} \frac{(n-1)!}{(v-1)!(n-v)!} p^{v-1} (1-p)^{n-v}$$

$$= np * \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} p^m (1-p)^{m-l} = np$$

[with a change of variable of $m = n-1$ and $l = v-1$]

$$E[V^2 - V] = \sum_{v=0}^{n} \frac{n!}{v!(n-v)!} (v^2 - v) p^v (1-p)^{n-v}$$

$$= n(n-1)p^2 * \sum_{v=2}^{n} \frac{(n-2)!}{(v-2)!(n-v)!} p^{v-2} (1-p)^{n-v}$$

$$= n(n-1)p^2 * \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} p^m (1-p)^{m-l} = n(n-1)p^2$$

$$\Rightarrow \text{Var[V]} = E[V^2 - V] + E[V] - (E[V])^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p);$$

(b) $S = V - (n - V) = 2V - n$, a linear function of a Gaussian $V$, when $n$ is large.

$\mu_S = E(S) = 2np - n = n(2p - 1)$, $\sigma_S^2 = \text{Var}[S] = 4\text{Var}[V] = 4np(1-p)$

Since $V$ is $N(np, np(1-p))$, $S$ is $N(n(2p-1), 4np(1-p))$;

(c) If we assume $p = 0.5$, then $f_S(s) = \frac{1}{\sqrt{2\pi} * 50} \exp[-\frac{s^2}{5000}]$, so

$$P(S < -300) = \frac{1}{\sqrt{2\pi} * 50} \int_{-\infty}^{-300} f_S(s) ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-6} e^{-x^2/2} dx \simeq 0$$

(d) Using the result in Part (c), the probability that you lost \$300 at the end of the game is almost zero, so it is highly likely that $p \neq 0.5$, i.e. the coin is biased. An unbiased estimate for $p$ is $\hat{p} = v/n$, with $v = (s+n)/2 = 1100$, or $\hat{p} = 0.44$;

(e) Using the result in Part(c), $V/n$ is $N(p, p(1-p)/n)$. So a 99% confidence interval can be established as $[\hat{p} - k\hat{\sigma}_p, \hat{p} + k\hat{\sigma}_p]$, such that $k = 2.575$, and $\hat{\sigma}_p \approx \sqrt{\hat{p}(1-\hat{p})/n} \approx 0.01 \Rightarrow$ an estimate of the 99% confidence interval is $[0.4143, 0.4658] \Rightarrow$ the value of $p = 0.5$ does not fall in the above interval;

(f) Base on the results in Parts (c), (d) and (e), we can conclude that the coin is not fair.

**Problem 4 (20%)**

It is desirable to estimate the difference between the mean sale prices for all residential properties sold during 1983 in two neighborhoods D and E of a mid-sized Florida city. A random set of $n_D$ sale prices for neighborhood D was collected with $X_{Di}$ being the *i*-th sample with unknown mean $\mu_D$ and variance $\sigma_D^2$. Another random set of $n_E$ sale prices for neighborhood E was also collected with $X_{Ej}$ being the *j*-th sample with unknown mean $\mu_E$ and variance $\sigma_E^2$. All the random samples are assumed to be mutually independent of each other. We are interested in estimating the mean difference $\mu_Y = \mu_D - \mu_E$.

(a) Define a new random variable *Y*, which serve as a good estimate for $\mu_Y$, if corresponding random samples are used to form a sample statistic for *Y*;

(b) Find the mean and standard deviation $\sigma_Y$ of *Y*, is *Y* an unbiased estimate?

(c) Give a sample statistic which serve as an unbiased point estimates for $\sigma_Y^2$;

(d) From the actual samples in surveys we have obtained the following set of measurements: $n_D = 30$ (sample size), $n_E = 40$ (sample size), $\bar{x}_D = \$52,356$ (sample mean), (sample mean), $\tilde{s}_{D2} = \$10,572$ (sample standard deviation), $\bar{x}_E = \$66,491$ (sample mean), and $\tilde{s}_{E2} = \$14,264$ (sample standard deviation). Construct a 95% confidence interval for $\mu_Y$ [Hint: Find the approximate large sample density for the corresponding sample statistic];

(e) If we want to test a null hypothesis, $H_0$: $\mu_Y = 0$ against an alternative hypothesis, $H_1$: $\mu_Y < 0$, construct a statistical test. Based on the data in part(c), decide if we should reject $H_0$ at a 95% confidence level, why?

**Solution:**

(a) Let $Y = \hat{\bar{X}}_D - \hat{\bar{X}}_E =$ sample mean difference between the two communities with $\hat{\bar{X}}_D = \frac{1}{n_D}\sum_{i=1}^{n_D} X_{Di}$ and $\hat{\bar{X}}_E = \frac{1}{n_E}\sum_{j=1}^{n_E} X_{Ej}$ where $x_{Di}$ and $x_{Ej}$ are samples.

(b) $E[Y] = E[\hat{\bar{X}}_D] - E[\hat{\bar{X}}_E] = \mu_D - \mu_E \Rightarrow Y$ is an unbiased estimate of the mean difference, and next we show in the following that $\sigma_Y = \sqrt{\frac{\sigma_D^2}{n_D} + \frac{\sigma_E^2}{n_E}}$ (intuitively sound).

$$\text{Var}[Y]=E[Y^2]-(E[Y])^2=\frac{1}{n_D^2}(\sum_{i=1}^{n_D}E[X_{Di}^2]+\sum_{i=1}^{n_D}\sum_{l\neq i}^{n_D}E[X_{Di}]E[X_{Di}])$$

$$+\frac{1}{n_E^2}(\sum_{j=1}^{n_E}E[X_{Ej}^2]+\sum_{j=1}^{n_E}\sum_{k\neq j}^{n_E}E[X_{Ej}]E[X_{Ek}])-\frac{2}{n_Dn_E}\sum_{i=1}^{n_D}\sum_{j=1}^{n_E}E[X_{Di}]E[X_{Ek}])-(\mu_D\text{-}\mu_E)^2$$

$$=\frac{1}{n_D^2}[n_D\sigma_D^2+n_D^2\mu_D^2]+\frac{1}{n_E^2}[n_E\sigma_E^2+n_E^2\mu_E^2]\text{-}\frac{2}{n_Dn_E}[(n_D\mu_D)(n_E\mu_E)]\text{-}(\mu_D\text{-}\mu_E)^2$$

$$=\frac{\sigma_D^2}{n_D}+\frac{\sigma_E^2}{n_E}$$ (sum of the two variances of the corresponding individual sample means);

(c) Let $\tilde{S}_D^2=\frac{1}{(n_D-1)}\sum_{i=1}^{n_D}E[(X_{Di}-\hat{\bar{X}}_D)^2]$ and $\tilde{S}_E^2=\frac{1}{(n_E-1)}\sum_{j=1}^{n_E}E[(X_{Ej}-\hat{\bar{X}}_E)^2]$, then

$E[\tilde{S}_{D2}^2]=\sigma_D^2$ and $E[\tilde{S}_{E2}^2]=\sigma_D^2\Rightarrow\ \tilde{S}_{D2}^2$ and $\tilde{S}_{E2}^2$ are unbiased estimates.

If we defined $S_Y^2=\frac{\tilde{S}_{D2}^2}{n_D}+\frac{\tilde{S}_{E2}^2}{n_E}$, then $S_Y^2$ is an unbiased estimate of $\sigma_{Y;}^2$

By replacing the above random variables with respective sample statistics,

we have a $S_Y^2$-statistic $=\frac{1}{n_D(n_D-1)}\sum_{i=1}^{n_D}E[(x_{Di}-\bar{x}_D)^2]+\frac{1}{n_E(n_E-1)}\sum_{j=1}^{n_E}E[(x_{Ej}-\bar{x}_E)^2]$;

(d) If we have large sample statistics, i.e. $n_D$ and $n_E$ are large enough (e.g. >=30) then $Y$ is the difference of two Gaussain random variables, so $Y$ is $N(\mu_Y,\sigma_Y^2)$.
An estimate of the 95% confidence is $[\bar{x}_D-\bar{x}_E-1.96s_Y,\bar{x}_D-\bar{x}_E+1.96s_Y]$, $s_Y=3095$, or [-\$20,201,-\$8,069], by plugging in the sample values from the given measurements;

(e) We can construct a one-sided test, by using the approximate Gaussain $z$-statistic,

$$z=\frac{(\bar{x}_D-\bar{x}_E)-\hat{\mu}_Y}{s_Y}=\frac{-14135}{3095}=-4.567\ll-1.645\ (\hat{\mu}_Y=0,s_Y=3{,}095,z_{0.05}=\pm1.645).$$

Therefore we can reject the hypothesis $H_0:\mu_Y$=0 at a 95% confidence level.

**Problem 5 (20%)**

A zero-mean random process *X(t)* is wide-sense stationary, with autocorrelation function $R_X(\tau)=10e^{-2|\tau|}-5e^{-4|\tau|}$. Another zero-mean, stationary noise process, *N(t)*, has a band-pass spectral density: $S_N(\omega)=1/5,\text{ for }10\pi\le|\omega|\le20\pi,\text{ and }S_N(\omega)=0\text{ elsewhere}$. Assume *X(t)* and *N(t)* are independent. Define a noisy process: *Y(t)=aX(t-q)+N(t)*, where *a* is a constant much smaller than 1, *q* represents a round-trip time delay in return signal acquisition to measure the distance to a target using the signal, *X(t)*.

(a) Find the spectral density, $S_X(\omega)$, and $S_X(s)$ (with $s\triangleq j\omega$) of process *X(t)*;
(b) Is $S_X(\omega)$ a rational spectrum? If so, find all the poles and zeroes of $S_X(s)$?
(c) Find $\int_{-\infty}^{\infty}S_X(\omega)d\omega$, and verify $\int_{-\infty}^{\infty}S_X(\omega)d\omega=R_X(0)=E[X^2]$ [Hint: the integration can be accomplished by the Residue Theorem];
(d) Find the mean-square value of the noise process, *N(t)*;
(e) Find the autocorrelation, $R_Y(\tau)$, and mean-square value, $R_Y(0)$, of *Y(t)* ;

(f) Compute the cross-correlation function, $R_{XY}(\tau)=E[X(t)Y(t+\tau)]$, and the cross-spectral density, $S_{XY}(\omega)=\int_{-\infty}^{\infty}R_{XY}(\tau)e^{-j\omega\tau}d\tau$;

(g) Describe a way to estimate the delay, $q$.

**Solution:**

(a) $S_X(\omega)=\int_{-\infty}^{\infty}R_X(\tau)e^{-j\omega\tau}d\tau=\int_{-\infty}^{0}10e^{(2-j\omega)\tau}d\tau+\int_{0}^{\infty}10e^{-(2+j\omega)\tau}d\tau-\int_{-\infty}^{0}5e^{(4-j\omega)\tau}d\tau-\int_{0}^{\infty}5e^{-(4+j\omega)\tau}d\tau$

$$=\frac{10e^{(2-j\omega)\tau}}{2-j\omega}\Big|_{-\infty}^{0}+\frac{-10e^{-(2+j\omega)\tau}}{2+j\omega}\Big|_{0}^{\infty}-\frac{5e^{(4-j\omega)\tau}}{4-j\omega}\Big|_{-\infty}^{0}-\frac{-5e^{-(4+j\omega)\tau}}{4+j\omega}\Big|_{-\infty}^{0}$$

$$=\frac{40}{4+\omega^2}-\frac{40}{16+\omega^2}=\frac{480}{(4+\omega^2)(16+\omega^2)}\Rightarrow S_X(s)=\frac{480}{(4-s^2)(16-s^2)};$$

(b) Clearly, $S_X(\omega)$ is a rational spectrum, and $S_X(s)$ has poles at $\pm 2,\ \pm 4$;

(c) $E[X^2]=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_X(\omega)d\omega=\text{residue}_{s=-2}+\text{residue}_{s=-4}$

$$=\frac{480}{-(2-s)(16-s^2)}\Big|_{s=-2}+\frac{480}{-(4-s)(4-s^2)}\Big|_{s=-4}=\frac{480}{48}-\frac{480}{96}=5;$$

It is also clear that $R_X(0)=10-5=5\Rightarrow E[X^2]=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_X(\omega)d\omega=R_X(0)=5;$

(d) For this ideal band-limited noise process, $\frac{1}{2\pi}\int_{-\infty}^{\infty}S_N(\omega)d\omega=2=E[N^2]$;

(e) $R_N(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_N(\omega)e^{j\omega\tau}d\omega=\frac{1}{10\pi}[\int_{-20\pi}^{-10\pi}e^{j\omega\tau}d\omega+\int_{10\pi}^{20\pi}e^{j\omega\tau}d\omega]$

$$=\frac{1}{10\pi}[\frac{e^{-j10\pi\tau}-e^{-j20\pi\tau}}{j\tau}+\frac{e^{j20\pi\tau}-e^{j10\pi\tau}}{j\tau}]=\frac{1}{5\pi}[\frac{\sin 20\pi\tau}{\tau}-\frac{\sin 10\pi\tau}{\tau}]=2[\frac{\sin 5\pi\tau}{5\pi\tau}]*\cos 15\pi\tau.$$

$$\Rightarrow R_Y(\tau)=a^2R_X(\tau-q)+R_N(\tau)=10a^2e^{-2|\tau-q|}-5^{a^2}e^{-4|\tau-q|}+2[\frac{\sin 5\pi\tau}{5\pi\tau}]*\cos 15\pi\tau$$

$$\Rightarrow R_Y(0)=10a^2e^{-2q}-5a^2e^{-4q}+2;$$

(f) $R_{XY}(\tau)=E[X(t)Y(t+\tau)]=aE[X(t)X(t+\tau-q)]+E[X(t)N(t+\tau-q)]$

$$=aE[X(t)X(t+\tau-q)]=aR_X(\tau-q)\Rightarrow S_{XY}(\omega)=aS_X(\omega)=\frac{480a}{64+20\omega+\omega^2};$$

(g) Since $R_{XY}(\tau)=aR_X(\tau-q)$ has a peak at $\tau=q\Rightarrow q$ can be estimated by evaluating $R_{XY}(\tau)$ from $X(t)$ and $Y(t)$, then doing peak picking over $\tau$.

**Problem 6 (Extra Credit: +20%)**

Consider a low pass RC circuit that characterize a linear and time-invariance (LTI) system, with a frequency response: *H(s)=b/(s+b)* with *b=1/RC* representing a positive decaying constant. It can be shown that the impulse response of the LTI system is simply $h(t)=be^{-bt},\ t\geq 0$, and $h(t)=0,\ t<0$. A random signal, *X(t)*, passes through the system will produce a random output, *Y(t)*, computed as the following convolution integral: $Y(t)=\int_{0}^{\infty}X(t-s)h(s)ds$ . In general *X(t)* is not zero mean.

(a) If *X(t)* is a white noise with $S_Y(f) = S_0, -\infty < f < \infty$, show that $R_X(\tau) = S_0\delta(\tau)$;

(b) Find $E[Y(t)]$ and $E[Y(t)Y(t+\tau)]$, and show that *Y(t)* is wide-sense stationary;

(c) If *X(t)* is non-white with $R_X(\tau) = Ke^{-\beta|\tau|}, \beta > 0$, find $S_Y(\omega)$. Determine the value of *K* such that $S_Y(0) = S_0$, i.e. same dc value as the above white noise case;

(d) Given the autocorrelation of *X(t)* in Part (c), find the autocorrelation of the corresponding system output, $R_Y(\tau) = E[Y(t)Y(t+\tau)]$. Verify that when $\beta \to \infty$, $R_Y(\tau)$ will converge to the solution in Part (b), an intuitive consequence;

(e) If $\beta \gg b$, i.e. the input spectrum has a much wider bandwidth than that (constant *b*) of the low pass LTI system response, show that the above white noise approximation in Part (d) still holds, another important result.

**Solution:**

(a) $R_X(\tau) = \lim_{F\to\infty} \int_{-F}^{F} S_0 e^{-j2\pi f\tau} df = \lim_{F\to\infty} 2FS_0 \dfrac{\sin 2F\tau}{2F\tau} = S_0\delta(\tau);$

(b) $E[Y(t)] = \int_0^\infty E[X(t-s)]be^{-bs} ds = E[X(t)]\int_0^\infty be^{-bs} ds = E[X(t)];$

$$E[Y(t)Y(t+\tau)] = \int_0^\infty \int_0^\infty E[X(t-s)X(t+\tau-u)]b^2 e^{-bs} e^{-bu} dsdu \text{ [for } \tau \ge 0]$$

$$= \int_0^\infty b^2 e^{-bs} ds \int_0^\infty R_X(\tau-u+s) e^{-bu} du = b^2 S_0 \int_0^\infty e^{-bs} e^{-b(s+\tau)} du = \frac{bS_0}{2} e^{-b\tau}$$

$\Rightarrow$ $Y(t)$ is a wide-sense stationary process;

(c) If $R_X(\tau) = Ke^{-\beta|\tau|} \Rightarrow S_X(\omega) = \dfrac{2K\beta}{\omega^2+\beta^2} \Rightarrow S_X(0) = \dfrac{2K}{\beta} = S_0 \Rightarrow K = \dfrac{\beta S_0}{2};$

(d) $E[Y(t)Y(t+\tau)] = \int_0^\infty b^2 e^{-bs} ds \int_0^\infty R_X(\tau-u+s) e^{-bu} du$ [For $\tau \ge 0$]

$$= \frac{b^2\beta S_0}{2}\{\int_0^\infty e^{-(b+\beta)s} ds * \int_0^{s+\tau} e^{-\beta\tau} e^{-(b-\beta)s} du\} \text{ [For } \tau \ge 0, u - s - \tau < 0]$$

$$+ \frac{b^2\beta S_0}{2}\{\int_0^\infty e^{-(b-\beta)s} ds * \int_{s+\tau}^\infty e^{\beta\tau} e^{-(b+\beta)s} du\} \text{ [For } \tau \ge 0, u - s - \tau \ge 0]$$

$$= \frac{b^2\beta S_0}{2}\{\frac{e^{-\beta\tau}}{-(b-\beta)}\int_0^\infty e^{-(b+\beta)s}[e^{-(b-\beta)(s+\tau)} - 1]ds + \frac{e^{\beta\tau}}{-(b+\beta)}\int_0^\infty e^{-(b-\beta)s} e^{-(b+\beta)(s+\tau)} ds\}$$

$$= \frac{b^2\beta S_0}{2} * \{\frac{1}{(b-\beta)}[-\frac{e^{-b\tau}}{2b} + \frac{e^{-\beta\tau}}{b+\beta}] + \frac{1}{(b+\beta)}\frac{e^{-b\tau}}{2b}\} = \frac{b^2\beta S_0}{2(b^2-\beta^2)} * [e^{-\beta\tau} - \frac{\beta e^{-b\tau}}{b}]$$

$\Rightarrow R_Y(\tau) = \dfrac{b^2\beta S_0}{2(b^2-\beta^2)} * [e^{-\beta|\tau|} - \dfrac{\beta e^{-b|\tau|}}{b}]$ $\Rightarrow Y(t)$ is a wide-sense stationary process;

If $\beta \to \infty,\ R_Y(\tau) \to \dfrac{bS_0}{2} e^{-b|\tau|}$ $\Rightarrow$ same autocorrelation as the white noise case in Part (b);

(e) $R_Y(\tau) = \dfrac{bS_0}{2} e^{-b|\tau|} * \dfrac{1}{1-(b^2/\beta^2)} * [1 - \dfrac{b}{\beta} e^{-(\beta-b|)\tau|}]$

If $\beta \gg b, R_X(\tau) \approx \dfrac{bS_0}{2} e^{-b|\tau|}$ $\Rightarrow$ approximating the white noise case.