Symplectic 4-Manifolds on the Noether Line and between the Noether and Half Noether Lines

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Introduction-Background

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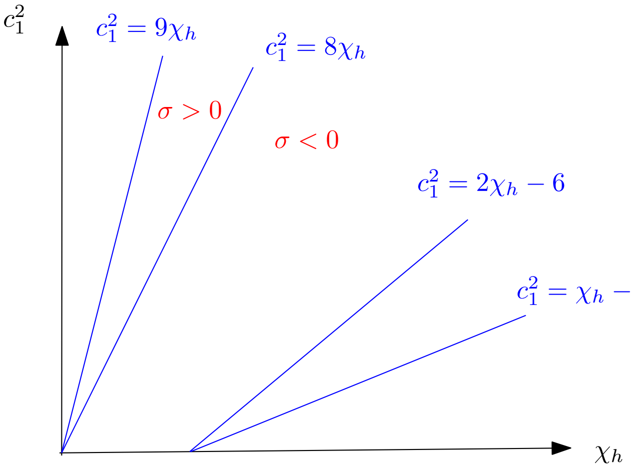
(Recall: For minimal complex surfaces S of general type, the BMY and Noether inequalities $9\chi_h(S) \geq c_1^2(S) \geq 2\chi_h(S) - 6$ hold.)

Geography Chart



Notations and Conventions

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  The I_n singular fiber, for $n \geq 2$, is a configuration of n 2-spheres of self intersections -2 arranged in a cycle and was given by Kodaira.

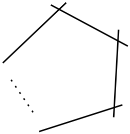

Figure: I_n fiber

Background on Seiberg-Witten invariants

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Recall that a simply connected smooth 4-manifold X is said to be *of simple type* if each basic class K satisfies the equation $K^2 = c_1^2(X) = 3\sigma(X) + 2e(X)$.

Now let us give a generalized blow-up formula.

Theorem (Fintushel-Stern)

Assume that a simply connected, smooth 4-manifold X' decomposes as $X' = X \# N$, where X is of simple type. If $b_2^+(N) = 0$ from where $H^2(N, \mathbb{Z})$ has an orthogonal basis $\{E_i \in H^2(N, \mathbb{Z}) \mid i = 1, 2, \cdots, b_2(N)\}$ with $E_i^2 = -1$, then $Bas_{X'} = \{K_i \pm E_1 \pm \cdots \pm E_{b_2(N)} \mid K_i \in Bas_X\}$.

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The basic classes of the elliptic surface $E(n)$, $n \geq 2$ are given as follows.

Theorem (Fintushel-Stern)

$Bas_{E(n)} = \{PD(k \cdot f) \in H^2(E(n), \mathbb{Z}) \mid k \equiv n \ (mod\ 2), |k| \leq n - 2\}$, $n \geq 2$,

where f is the homology class of the fiber of $E(n)$ and PD means taking the Poincaré dual of the homology class.

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Also, Starkston showed that infinitely many star surgeries are not equivalent to any sequences of generalized symplectic rational blowdowns.

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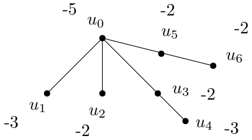
$\mathcal{Q}$ is the configuration of symplectic spheres which intersect according to a star shaped graph with 4 arms:



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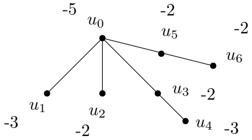

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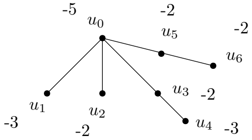

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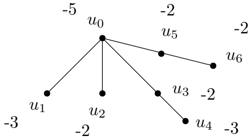

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Definition (Karakurt-Starkston)

Replacing the neighborhood of $\mathcal{Q}$ in a symplectic 4-manifold by the filling $\mathcal{R}$ is called the $(\mathcal{Q}, \mathcal{R})$ surgery.

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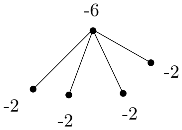

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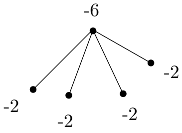

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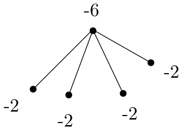

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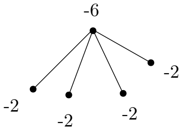

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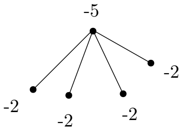

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$\mathcal{T}_2$ is a particular symplectic 4-manifold with Euler characteristic 3, $\pi_1(\mathcal{T}_2) = \mathbb{Z}/2$, $\sigma(\mathcal{T}_2) = -2$.

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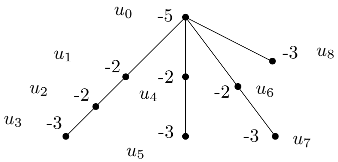

Figure: The configuration $\mathcal{U}$

The symplectic filling $\mathcal{V}$ is a particular symplectic 4-manifold with $e(\mathcal{V}) = 3$, $\sigma(\mathcal{V}) = -2$.

Definition (Karakurt-Starkston)

The $(\mathcal{U}, \mathcal{V})$ surgery is symplectically replacing the neighborhood of $\mathcal{U}$ in a symplectic 4-manifold by the filling $\mathcal{V}$.

Our Constructions

Theorem (S.)

There exist simply connected, minimal, symplectic 4-manifolds, each with an exotic smooth structure, and with one SW basic class up to sign, lying on the Noether line and between the Noether and half Noether lines obtained by star surgeries and by using complex singularities.

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We have also constructed configurations which consist of more than one I_n type complex singularities in the rational elliptic surfaces, geometrically (without using any monodromy arguments).

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Theorem (Miranda, Persson)

Every Jacobian rational elliptic surface is the blow up of the basepoints of a cubic pencil.

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Naruki explicitly constructed pencils of cubic curves in $\mathbb{CP}^2$. Later, Kurumadani generalized Naruki's work; more cubic pencils were shown to exist. But it is not shown how to obtain I_n configurations with $n \geq 2$ from the given pencils.

In our work, we have explicitly constructed the following configurations in $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$:

$$(I_6, I_3, I_2)$$
$$(I_5, I_4)$$
$$(I_5, I_5)$$

where the notation (I_6, I_3, I_2) means there is one singular fiber of type I_6, one of type I_3, and one of type I_2.

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where the notation (I_6, I_3, I_2) means there is one singular fiber of type I_6, one of type I_3, and one of type I_2. We have found the homology classes of the sphere components of each fiber, verified that their self intersections are -2 and precisely obtained the -1 sections.

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We start with the pencil P in $\mathbb{CP}^2$ given by Naruki. P is generated by the two cubic curves $L \cup Q$, and C with base points $p = (0, 0, 1)$ and $q = (1, 0, 0)$. C has a node at p. Then, he gives a member C_1 of P, which has a node at q and passing through p.

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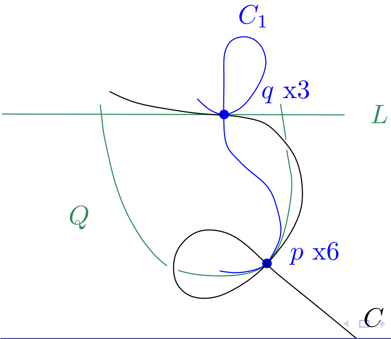

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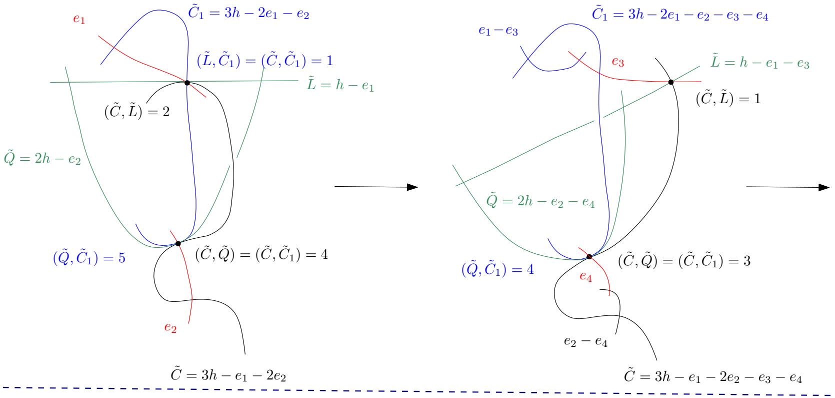


Figure: Construction of the (I_6, I_3, I_2) configuration in $E(1)$

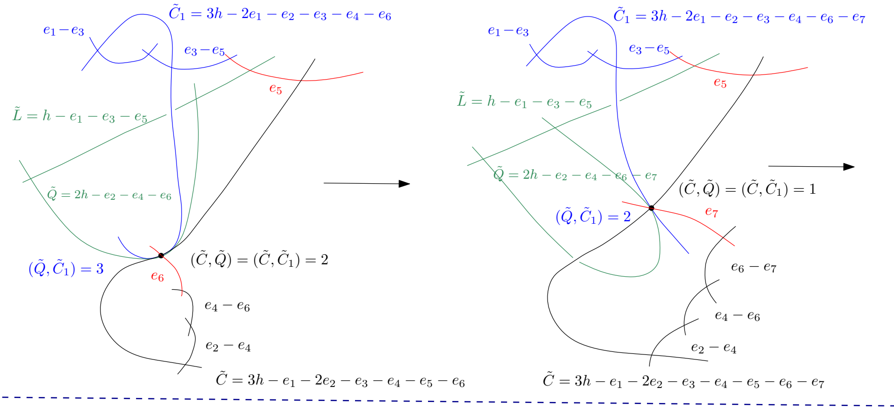
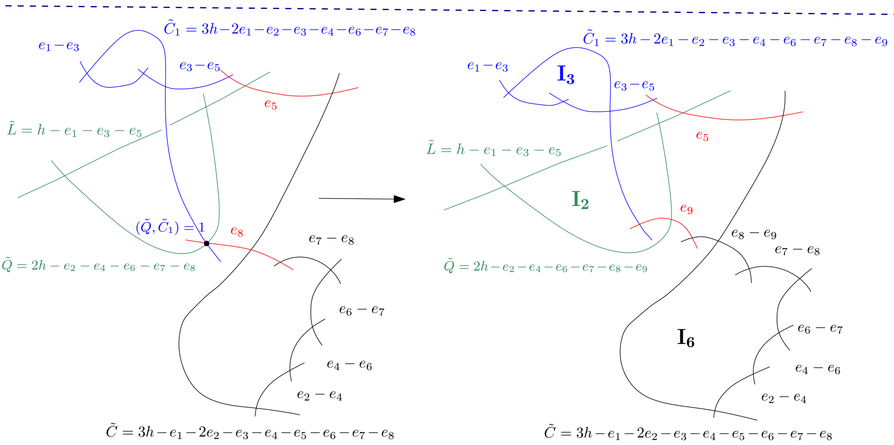


Figure: Cont'd: Construction of the (I_6, I_3, I_2) configuration in $E(1)$



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2. Construction of the (l_5, l_4) configuration in $E(1)$

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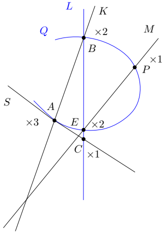

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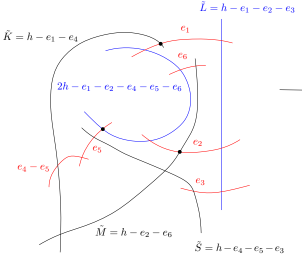

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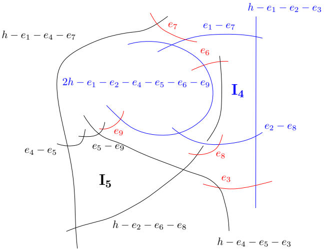

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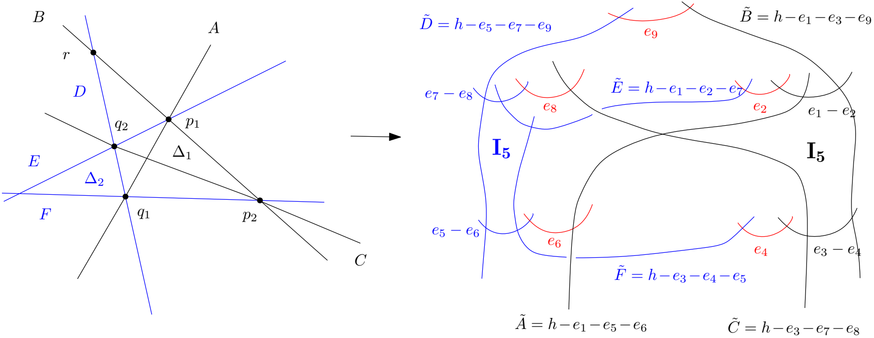

Figure: Construction of the (I_5, I_5) configuration in $E(1)$

Constructions of the plumbings $\mathcal{Q}, \mathcal{K}, \mathcal{S}_2, \mathcal{U}$ from the $(l_6, l_3, l_2), (l_5, l_4)$ and (l_5, l_5) configurations

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We prove each of the following lemmas in three different ways, from the $(l_6, l_3, l_2), (l_5, l_4)$ and (l_5, l_5) configurations that we constructed above:

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Lemma (S.)

The plumbing $\mathcal{Q}$ symplectically embeds in $E(5)\#\overline{\mathbb{CP}}^2$.

Constructions of the plumbings $\mathcal{Q}, \mathcal{K}, \mathcal{S}_2, \mathcal{U}$ from the $(l_6, l_3, l_2), (l_5, l_4)$ and (l_5, l_5) configurations

We prove each of the following lemmas in three different ways, from the $(l_6, l_3, l_2), (l_5, l_4)$ and (l_5, l_5) configurations that we constructed above:

Lemma (S.)

The plumbing $\mathcal{Q}$ symplectically embeds in $E(5)\#\overline{\mathbb{CP}}^2$.

Lemma (S.)

The plumbing $\mathcal{K}$ symplectically embeds in $E(6)$, and the plumbing $\mathcal{S}_2$ symplectically embeds in $E(5)$.

Constructions of the plumbings $\mathcal{Q}, \mathcal{K}, \mathcal{S}_2, \mathcal{U}$ from the $(l_6, l_3, l_2), (l_5, l_4)$ and (l_5, l_5) configurations

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Lemma (S.)

The plumbing $\mathcal{U}$ symplectically embeds in $E(5)\#3\overline{\mathbb{CP}}^2$.

Constructions of simply connected, minimal, symplectic and exotic 4-manifolds on the Noether line

Constructions of simply connected, minimal, symplectic and exotic 4-manifolds on the Noether line

Theorem (S.)

There exists a simply connected, minimal, symplectic 4-manifold X with an exotic smooth structure, and with one SW basic class up to sign, lying on the Noether line and obtained by the $(\mathcal{Q}, \mathcal{R})$ star surgery.

Constructions of simply connected, minimal, symplectic and exotic 4-manifolds on the Noether line

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where

$$X := ((E(5)\#\overline{\mathbb{CP}}^2) \setminus \mathcal{Q}) \cup \mathcal{R}$$

$$\chi_h(X) = 5 \quad \text{and} \quad c_1^2(X) = 4 = 2\chi_h - 6 \tag{2}$$

Constructions of simply connected, minimal, symplectic and exotic 4-manifolds on the Noether line

Minimality:

- We first check which basic classes of $E(5)\#\overline{\mathbb{CP}}^2$ extends to X.

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- By using the dimension of the Seiberg-Witten moduli space we show that $f + E_1$, $f - E_1$ and $3f - E_1$ do not descend to X as a basic class.

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- By using the dimension of the Seiberg-Witten moduli space we show that $f + E_1$, $f - E_1$ and $3f - E_1$ do not descend to X as a basic class.
- But, up to sign, the last class $M := 3f + E_1$ extends to the symplectic manifold X as a basic class (by Taubes' result.)

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- But, up to sign, the last class $M := 3f + E_1$ extends to the symplectic manifold X as a basic class (by Taubes' result.)
- Next we prove that the class M extends to X uniquely and X has one basic class up to sign.

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- By using the dimension of the Seiberg-Witten moduli space we show that $f + E_1$, $f - E_1$ and $3f - E_1$ do not descend to X as a basic class.
- But, up to sign, the last class $M := 3f + E_1$ extends to the symplectic manifold X as a basic class (by Taubes' result.)
- Next we prove that the class M extends to X uniquely and X has one basic class up to sign. By the blow up formula, X is minimal.

Constructions of simply connected, minimal, symplectic and exotic 4-manifolds on the Noether line

Theorem (S.)

There exists a simply connected, minimal, symplectic 4-manifold T with an exotic smooth structure, and with one SW basic class up to sign, lying on the Noether line. T is obtained by the $(\mathcal{U}, \mathcal{V})$-star surgery and homeomorphic to the manifold X above.

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X and T are exotic copies of $9\mathbb{CP}^2 \# 45\overline{\mathbb{CP}}^2$.

Constructions of simply connected, minimal, symplectic and exotic 4-manifolds between the Noether and half Noether lines

Theorem (S.)

There exists a simply connected, minimal, symplectic 4-manifold Y with an exotic smooth structure, and with one SW basic class up to sign, lying in between the Noether and the half Noether lines, obtained by the $(\mathcal{K}, \mathcal{L})$*-star surgery.*

Y is an exotic copy of $11\mathbb{CP}^2\#55\overline{\mathbb{CP}}^2$.

Constructions of simply connected, minimal, symplectic and exotic 4-manifolds between the Noether and half Noether lines

Theorem (S.)

There exists a simply connected, minimal, symplectic 4-manifold Y with an exotic smooth structure, and with one SW basic class up to sign, lying in between the Noether and the half Noether lines, obtained by the $(\mathcal{K}, \mathcal{L})$-star surgery.

Y is an exotic copy of $11\mathbb{CP}^2\#55\overline{\mathbb{CP}}^2$.

Theorem (S.)

There exists a simply connected, minimal, symplectic 4-manifold Z with an exotic smooth structure, and with one SW basic class up to sign, lying in between the Noether and the half Noether lines, obtained by the $(\mathcal{S}_2, \mathcal{T}_2)$-star surgery.

Z is an exotic copy of $9\mathbb{CP}^2\#46\overline{\mathbb{CP}}^2$.

A simply connected, minimal, symplectic and exotic 4-manifold above the Noether line

Theorem (S.)

There exists a simply connected, minimal, symplectic 4-manifold M with an exotic smooth structure, and with one SW basic class up to sign, lying above the Noether line and obtained by the $(\mathcal{U}, \mathcal{V})$*-star surgery.*

M is an exotic copy of $3\mathbb{CP}^2 \# 17\overline{\mathbb{CP}}^2$.

THANKS!