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Vibration mode structure and simplified modelling of cyclically symmetric or rotationally periodic systems

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This paper proves the modal vibration properties of general rotating, cyclically symmetric (or rotationally periodic) systems, including those with central components to which cyclically symmetric substructures are attached. This cyclic symmetry results in structured modal properties with only two possible mode types referred to as substructure and coupled modes. For systems with uncoupled central component translations and rotations, which is the usual case, all eigenvectors fall into one of three categories: substructure, translational and rotational modes. The properties of the system equations of motion resulting from the cyclic symmetry are discussed first. These properties are then used to prove the modal decomposition of general rotating, cyclically symmetric systems. The development leads to modelling and computational efficiencies. The vibration modes and natural frequencies for each mode type are determined from reduced eigenvalue problems that are much smaller than the full system eigenvalue problem. The full system matrices, although not needed for the current purposes, can be generated from much smaller matrices derived from simple subsystem models.

1. Introduction

Many mechanical and electrical devices are cyclically symmetric (or rotationally periodic), including turbine discs, helicopter rotors, turboprop assemblies, planetary gears, various spinning disc applications (e.g. slotted discs [1], disc–spindle systems [2,3] and discs with attachments [4,5]) and circular rings [6,7]. Such systems have identical substructures that are equally spaced circumferentially. Cyclically symmetric systems possess unique modal properties that are derived and proved in this paper. Knowledge of these properties can significantly simplify analysis of the vibration response and control of these systems. Fortescue [8] was among the earliest to systematically study cyclic symmetry, which has been adopted in modern finite-element method [9].

We divide the class of cyclically symmetric systems into cases with and without central component vibrations. In cases where the central components do not vibrate (although they may rotate at prescribed speeds), the substructures are connected to other substructures and to the central components. Shen [10] studied such systems using a transfer function formulation. The mass and stiffness matrices of these systems are circulant or block circulant [11,12]. Orris & Petyt [13] and Thomas [14,15] studied wave propagation in these systems, which shows the phase relations between the substructures. These phase relations can be characterized by phase indices defined by Kim *et al.* [2,3]. Óttarsson [16], Olson [17] and Bladh [18] investigated bladed discs in turbomachinery as a cyclically symmetric system without central component vibrations. Circulant or block circulant matrix formulations are obtained in these systems.

In the more general case of cyclically symmetric systems, central component vibrations are considered. In this case, the substructures are attached to one or more oscillating central component that can also rotate at prescribed speeds. The substructures can also connect to other substructures. The circumferentially uniform central components are axisymmetric about the system's central axis. Planetary gears [6,19–22] and centrifugal pendulum vibration absorber (CPVA) systems [23–25] are two examples of this case. The sun gear, ring gear and carrier are the central components of a planetary gear, and the planet gears are the substructures. The rotor is the central component of a CPVA system, and the pendulum absorbers are the substructures. Past works analysing these specific systems have categorized their vibration modes into three categories: rotational, translational and planet/absorber modes [6,19–25]. These studies, however, deal only with the equations of motion for the two specific systems. No generalizations were attempted, and they are not apparent from the past research. Most features of the vibration modes for these two systems remain when the substructures are in diametrically opposed pairs rather than equally spaced [24,26,27]. This eigensolution decomposition is valuable in related analytical studies on the vibration of these systems [28–33].

Kim *et al.* [2,3] discovered that, in disc–spindle systems, the modes with phase index $k = 0, 1,$ and $N - 1$ exert either a net torque or force on the central components (N is the number of substructures). These modes associate with the rotational and translational modes in planetary gears and CPVA systems [6,19–25]. All the other modes with phase index $k = 2, 3, \dots, N - 2$ do not exert a net torque or force on the central components. These correspond to substructure modes in this work. Planet/absorber modes in planetary gears and CPVA systems are examples of substructure modes.

The modal properties of a general rotating, cyclically symmetric system are derived in this paper. Cyclically symmetric systems with non-vibrating central components are studied first, demonstrating unique characteristics of the system eigensolutions. These characteristics are then used to analyse the modal properties of cyclically symmetric systems with central component vibrations. Substructure modes with pure substructure motions and coupled modes that contain both central component and substructure motions form the vibration mode structure of these systems. The vast majority of systems have uncoupled central component translations and rotations; for such systems, the eigenvectors are further categorized as substructure, translational and rotational modes.

We are not aware of prior work analysing the vibration of cyclically symmetric systems with central components that can vibrate in both translation and rotation. The present model allows for rotation of all components (i.e. gyroscopic effects) as well as asymmetry of the stiffness and damping matrices. The main contributions of this work are: (i) the mathematical derivation and characterization of the highly structured modal properties of such systems, (ii) the definition of reduced eigenvalue problems (involving matrices that are much smaller than the full system ones) that yield the complete set of eigensolutions for the full system with far less computational expense, and (iii) the identification of three subsystem models from which the full system equations are readily constructed, thus minimizing the effort of modelling the full system; these subsystem models involve only the central components, one substructure and any substructures directly connected to this one substructure.

2. Cyclically symmetric systems with non-vibrating central components

A cyclically symmetric system with non-vibrating central components consists of N equally spaced, identical substructures attached to central components (figure 1). The central components do not vibrate relative to their nominal position, although they may rotate at specified constant speeds. The substructures connect with each other. The reference frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ shown in figure 1 has $\mathbf{e}_3$ oriented along the system axis of symmetry; it can rotate about this axis at a constant speed that is the rotation speed of the substructures. The basis $\{\mathbf{e}_1^{(i)}, \mathbf{e}_2^{(i)}, \mathbf{e}_3^{(i)}\}$ associated with the ith substructure has a fixed angular orientation relative to the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ reference frame. The constant angle β_i between $\mathbf{e}_1$ and $\mathbf{e}_1^{(i)}$ defines the circumferential position of the ith substructure relative to the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ reference frame. For convenience, the angles are assigned such that $\beta_1 = 0$ and $\beta_i = 2(i-1)\pi/N$.

Each substructure has L degrees of freedom, and these motions can be three dimensional. The L degrees of freedom (or generalized coordinates) for each of two arbitrarily selected ith and jth substructures are identical when viewed by observers fixed to each of the $\{\mathbf{e}_1^{(i)}, \mathbf{e}_2^{(i)}, \mathbf{e}_3^{(i)}\}$ and $\{\mathbf{e}_1^{(j)}, \mathbf{e}_2^{(j)}, \mathbf{e}_3^{(j)}\}$ bases. In other words, the degree-of-freedom definitions in each substructure preserve the cyclic symmetry of the system.

The eigenvalue problem of the system is

$$\lambda^2 \mathbf{M}_\mathrm{s}\mathbf{u} + \lambda(\mathbf{G}_\mathrm{s} + \mathbf{C}_\mathrm{s})\mathbf{u} + (\mathbf{K}_\mathrm{s} + \mathbf{H}_\mathrm{s})\mathbf{u} = \mathbf{A}_\mathrm{s}\mathbf{u} = \mathbf{0} \tag{2.1a}$$

and

$$\mathbf{A}_\mathrm{s} = \lambda^2 \mathbf{M}_\mathrm{s} + \lambda(\mathbf{G}_\mathrm{s} + \mathbf{C}_\mathrm{s}) + \mathbf{K}_\mathrm{s} + \mathbf{H}_\mathrm{s}, \tag{2.1b}$$

where $\mathbf{M}_\mathrm{s}$, $\mathbf{G}_\mathrm{s}$, $\mathbf{C}_\mathrm{s}$, $\mathbf{K}_\mathrm{s}$ and $\mathbf{H}_\mathrm{s}$ are, respectively, the mass, gyroscopic, damping, stiffness and circulatory matrices. Because the central components do not vibrate, they do not affect the system equations of motion. The $NL \times 1$ vector $\mathbf{u}$ contains only the substructure degrees of freedom.

As discussed by Óttarsson [16] and Olson [17], the mass and stiffness matrices $\mathbf{M}_\mathrm{s}$ and $\mathbf{K}_\mathrm{s}$ are block circulant, where a block circulant matrix $\mathbf{B}$ is an $NL \times NL$ matrix formed by L-dimensional submatrices $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_N$ such that [11]

$$\mathbf{B} = \mathrm{circulant}(\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_N) = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \cdots & \mathbf{B}_N \\ \mathbf{B}_N & \mathbf{B}_1 & \cdots & \mathbf{B}_{N-1} \\ \vdots & \vdots & \ddots & \cdots \\ \mathbf{B}_2 & \mathbf{B}_3 & \cdots & \mathbf{B}_1 \end{pmatrix}. \tag{2.2}$$

Óttarsson [16] and Olson [17] only considered the mass and stiffness operators, but this block circulant property also holds for the operator $\mathbf{A}_\mathrm{s}$. The equations that govern the first substructure's motion yield a matrix operator derived from the first L rows of equation (2.1) as

$$\mathbf{A}_\mathrm{s}^{(1)} = (\mathbf{A}_1 \quad \mathbf{A}_2 \quad \cdots \quad \mathbf{A}_N), \tag{2.3}$$

where the $\mathbf{A}_i$ are $L \times L$ matrices. $\mathbf{A}_1$ describes the effect of the first substructure on itself, whereas $\mathbf{A}_i$ for $i = 2, 3, \ldots, N$ describes the effect of the ith substructure on the first substructure. Because of these physical meanings of the matrices $\mathbf{A}_i$ for $i = 1, 2, \ldots, N$, the cyclic symmetry of the system and the aforementioned stipulation that the generalized coordinate definitions for each

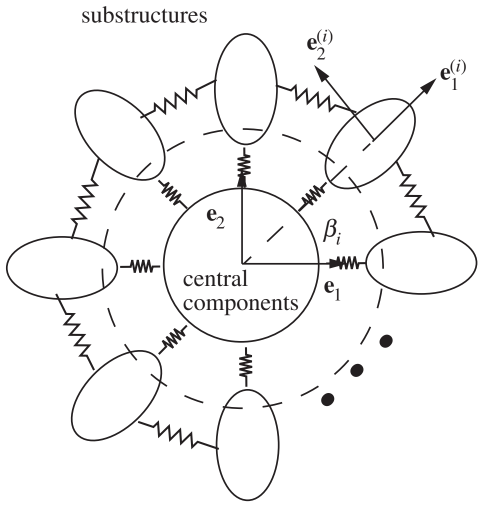

Figure 1. The configuration of cyclically symmetric systems. The solid and dashed circles represent the possible central components. The substructures are represented by the ovals. The reference frame and the basis fixed to the *i*th substructure are shown.

substructure be the same when viewed in the local reference frame for each substructure, the equations of motion for the second substructure yield the $L \times NL$ matrix operator

$$\mathbf{A}_s^{(2)} = (\mathbf{A}_N \quad \mathbf{A}_1 \quad \cdots \quad \mathbf{A}_{N-1}). \tag{2.4}$$

After completing a circle for all the N substructures, the block circulant matrix operator $\mathbf{A}_s$ is obtained with the $\mathbf{A}_s^{(i)}$ stacked as its rows.

The vectors

$$\left.\begin{aligned} \mathbf{u}_{kl} &= (\mathrm{e}^{jk\beta_1}\mathbf{v}_{kl}, \mathrm{e}^{jk\beta_2}\mathbf{v}_{kl}, \ldots, \mathrm{e}^{jk\beta_N}\mathbf{v}_{kl})^{\mathrm{T}} \\ \text{and} \qquad k &= 0, 1, \ldots, N-1, \quad l = 1, 2, \ldots, L, \end{aligned}\right\} \tag{2.5}$$

where j is the imaginary unit and $\mathbf{v}_{kl}$ are L-dimensional vectors determined below, are a set of NL independent eigenvectors for block circulant matrices as proved by Davis [11], Óttarsson [16] and Olson *et al.* [12]. Substitution of $\mathbf{u}_{kl}$ in equation (2.5) for a chosen k into the equation that governs the motion of each substructure in equation (2.1) (that is, the L equations defined by each $\mathbf{A}_s^{(i)}$) all yield

$$\left(\sum_{i=1}^{N} \mathrm{e}^{jk\beta_i}\mathbf{A}_i\right)\mathbf{v}_{kl} = \mathbf{0}, \quad k = 0, 1, \ldots, N-1, \tag{2.6}$$

because one can multiply the matrix equation that governs the ith substructure motion by $\mathrm{e}^{-jk\beta_i}$, which yields equation (2.6). Each of the N equations (one for each $k = 0, 1, \ldots, N-1$) in equation (2.6) is an $L \times L$ eigenvalue problem that yields L independent reduced eigenvectors $\mathbf{v}_{kl}$ with associated eigenvalues λ_{kl}. Substitution of these reduced eigenvectors $\mathbf{v}_{kl}$ into equation (2.5), along with the known eigenvalues λ_{kl} from equation (2.6), generates NL eigensolutions $(\lambda_{kl}, \mathbf{u}_{kl})$ for the eigenvalue problem in equation (2.1). The number of eigensolutions obtained matches the dimension of the eigenvalue problem in equation (2.1). Thus, the eigensolutions $(\lambda_{kl}, \mathbf{u}_{kl})$ for $k = 0, 1, \ldots, N-1$ and $l = 1, 2, \ldots, L$ complete the entire eigenspace of equation (2.1).

The eigenvectors in equation (2.5) show that all vibration modes of a cyclically symmetric system with non-vibrating central components consist of identical motions for each substructure (i.e. the $\mathbf{v}_{kl}$), but the substructures vibrate out of phase with the phase difference determined by the integer k, known as the phase index of the vibration mode [2,3,23–25], and the spacing angles β_i.

The vectors $\mathbf{v}_{kl}$ are complex for gyroscopic, circulatory and/or damped systems, so the L degrees of freedom within a given substructure do not move in phase with each other. Nevertheless, the motions of the same degrees of freedom on any two substructures (say the ith and jth ones) are the same with a phase difference $k(\beta_j - \beta_i)$ determined by the complex exponential factors in equation (2.5).

3. Cyclically symmetric systems with central component motions

In a cyclically symmetric system with central component vibrations, the previously discussed substructures are connected to P axisymmetric, oscillating central components that may rotate at constant speeds, as shown in figure 1. The bases previously defined in figure 1 for the case without central component motions are still used in the present analysis. The reference frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ rotates with the substructures at a constant reference rotation speed Ω. If multiple central components are present, each one can rotate at a different speed. All these rotation speeds can be expressed as known multiples of the rotation speed of one of the central components, that is, the chosen reference rotation speed Ω. In planetary gears as an example, the sun gear, ring gear and carrier rotate at different speeds, but these speeds are calculated from the rotation speed of one reference frame (e.g. the carrier) multiplied by the gear ratios [6,19–22]. Some central components might rotate while others might be stationary. A planetary gear, for example, has rotating and stationary central components. Some central components can vibrate while others might admit no vibration. While the substructures can have three-dimensional motion, each central component is restricted to planar vibration with two translational degrees of freedom and one rotational degree of freedom about the axis of symmetry shared by all central components.

The system eigenvalue problem is

$$\lambda^2 \mathbf{M}\boldsymbol{\phi} + \lambda(\mathbf{G} + \mathbf{C})\boldsymbol{\phi} + (\mathbf{K} + \mathbf{H})\boldsymbol{\phi} = \mathbf{0} \tag{3.1a}$$

and

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_\mathrm{c} & \mathbf{M}_\mathrm{cs} \\ \mathbf{M}_\mathrm{cs}^\mathrm{T} & \mathbf{M}_\mathrm{s} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \mathbf{G}_\mathrm{c} & \mathbf{G}_\mathrm{cs} \\ -\mathbf{G}_\mathrm{cs}^\mathrm{T} & \mathbf{G}_\mathrm{s} \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \mathbf{C}_\mathrm{c} & \mathbf{C}_\mathrm{cs} \\ \mathbf{C}_\mathrm{cs}^\mathrm{T} & \mathbf{C}_\mathrm{s} \end{pmatrix},$$

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_\mathrm{c} & \mathbf{K}_\mathrm{cs} \\ \mathbf{K}_\mathrm{cs}^\mathrm{T} & \mathbf{K}_\mathrm{s} \end{pmatrix} \quad \text{and} \quad \mathbf{H} = \begin{pmatrix} \mathbf{H}_\mathrm{c} & \mathbf{H}_\mathrm{cs} \\ -\mathbf{H}_\mathrm{cs}^\mathrm{T} & \mathbf{H}_\mathrm{s} \end{pmatrix}. \tag{3.1b}$$

The subscripts c and s indicate that the submatrices associate with the central components, substructures or both of them. The eigenvector $\boldsymbol{\phi}$ associated with the eigenvalue λ is

$$\boldsymbol{\phi} = (\boldsymbol{\phi}_\mathrm{c}, \boldsymbol{\phi}_\mathrm{s})^\mathrm{T}, \tag{3.2a}$$

$$\boldsymbol{\phi}_\mathrm{c} = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_P)^\mathrm{T} \tag{3.2b}$$

and

$$\boldsymbol{\phi}_p = (x_p, y_p, \mu_p), \quad p = 1, 2, \ldots, P. \tag{3.2c}$$

The coordinates x_p and y_p are the central component translational vibrations along $\mathbf{e}_1$ and $\mathbf{e}_2$, and μ_p is the rotational vibration about the system's central axis. $\boldsymbol{\phi}_\mathrm{s}$ is an NL-dimensional vector containing all substructure degrees of freedom. As above, for notational simplicity, we define the following operator and eigenvalue problem equivalent to equation (3.1):

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_\mathrm{c} & \mathbf{A}_\mathrm{cs} \\ \mathbf{A}_\mathrm{sc} & \mathbf{A}_\mathrm{s} \end{pmatrix} = \lambda^2 \mathbf{M} + \lambda(\mathbf{G} + \mathbf{C}) + \mathbf{K} + \mathbf{H} \tag{3.3a}$$

and

$$\mathbf{A}\boldsymbol{\phi} = \mathbf{0}. \tag{3.3b}$$

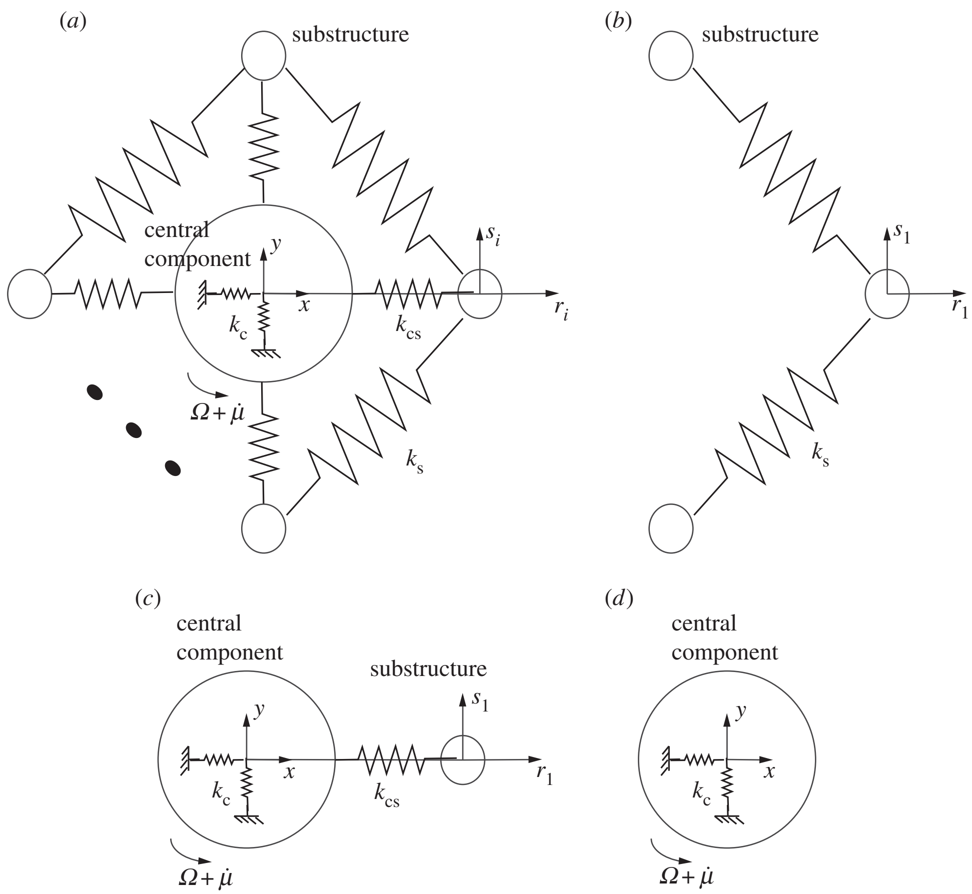

Figure 2. Example cyclically symmetric system with one central component and N substructures is shown in (*a*). Simplified models for obtaining $\mathbf{A}_s$, $\mathbf{A}_{cs}$ and $\mathbf{A}_c$ in equation (3.3*a*) are shown in (*b*–*d*), respectively.

(a) An example system with cyclic symmetry

Before investigating the properties of general cyclically symmetric systems with central component vibrations, the equations of motion for an example system are given in this section. This example will be referenced subsequently. It possesses all the properties of our general cyclically symmetric systems. Planetary gears [6,19–22] and CPVA systems [23–25] are other examples, but their equations of motion are lengthier and do not involve direct connections between substructures.

The example system is shown in figure 2. This example system contains a central component with N identical and equally spaced point masses connected to it. The spacing of the masses is determined by β_i. The central component rotates at a constant mean speed Ω and has two translational degrees of freedom (x and y relative to the body-fixed basis rotating at speed Ω) and one rotational degree of freedom (μ). The ith point mass has a radial degree of freedom r_i and a tangential degree of freedom s_i. Each substructure is connected to the central component by a stiffness k_{cs}. Neighbouring substructures are connected by a stiffness k_s. The distance between the steady deflection position of each mass (for rotation speed Ω) and the central axis is l. Both r_i and s_i are displacements from the steady deflection position. The isotropic central component bearing stiffness is k_c. The central component has mass m_c and moment of inertia J_c. Each substructure has mass m.

The linearized equations of motion are

$$\mathbf{M}\ddot{\mathbf{q}} + \Omega\mathbf{G}\dot{\mathbf{q}} + (\mathbf{K}_V - \Omega^2\mathbf{K}_T)\mathbf{q} = \mathbf{F}, \tag{3.4a}$$

$$\mathbf{q} = (x, y, \mu, r_1, s_1, r_2, s_2, \ldots, r_N, s_N)^{\mathrm{T}}, \tag{3.4b}$$

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_c & \mathbf{M}_{cs} \\ \mathbf{M}_{cs}^{\mathrm{T}} & \mathbf{M}_s \end{pmatrix}, \tag{3.4c}$$

$$\mathbf{G} = 2\begin{pmatrix} \mathbf{G}_c & \mathbf{G}_{cs} \\ -\mathbf{G}_{cs}^{\mathrm{T}} & \mathbf{G}_s \end{pmatrix}, \tag{3.4d}$$

$$\mathbf{K}_V = \begin{pmatrix} \mathbf{K}_{vc} & \mathbf{0}_{3\times 2N} \\ \mathbf{0}_{2N\times 3} & \mathbf{K}_{vs} \end{pmatrix} \tag{3.4e}$$

and

$$\mathbf{K}_T = \begin{pmatrix} \mathbf{K}_c & \mathbf{K}_{cs} \\ \mathbf{K}_{cs}^{\mathrm{T}} & \mathbf{K}_s \end{pmatrix}. \tag{3.4f}$$

The submatrices are

$$\mathbf{M}_c = \mathrm{diag}(m_c + Nm, m_c + Nm, J_c + Nml^2), \tag{3.5a}$$

$$\mathbf{M}_{cs} = m\begin{pmatrix} \cos\beta_1 & -\sin\beta_1 & \cos\beta_2 & -\sin\beta_2 & \cdots & \cos\beta_N & -\sin\beta_N \\ \sin\beta_1 & \cos\beta_1 & \sin\beta_2 & \cos\beta_2 & \cdots & \sin\beta_N & \cos\beta_N \\ 0 & l & 0 & l & \cdots & 0 & l \end{pmatrix}, \tag{3.5b}$$

$$\mathbf{M}_s = \mathrm{diag}(\underbrace{m, m, \ldots, m}_{2N}), \tag{3.5c}$$

$$\mathbf{G}_c = (m_c + Nm)\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.5d}$$

$$\mathbf{G}_{cs} = m\begin{pmatrix} -\sin\beta_1 & -\cos\beta_1 & -\sin\beta_2 & -\cos\beta_2 & \cdots & -\sin\beta_N & -\cos\beta_N \\ \cos\beta_1 & -\sin\beta_1 & \cos\beta_2 & -\sin\beta_2 & \cdots & \cos\beta_N & -\sin\beta_N \\ -l & 0 & -l & 0 & \cdots & -l & 0 \end{pmatrix}, \tag{3.5e}$$

$$\mathbf{G}_s = \mathrm{diag}(\underbrace{\hat{\mathbf{G}}_s, \hat{\mathbf{G}}_s, \ldots, \hat{\mathbf{G}}_s}_{N}), \tag{3.5f}$$

$$\hat{\mathbf{G}}_s = m\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.5g}$$

$$\mathbf{K}_{vc} = \mathrm{diag}(k_c, k_c, 0), \tag{3.5h}$$

$$\mathbf{K}_{vs} = \mathrm{circulant}\left(\mathbf{K}_1, \mathbf{K}_2, \underbrace{\mathbf{0}_{2\times 2}, \mathbf{0}_{2\times 2}, \ldots, \mathbf{0}_{2\times 2}}_{N-3}, \mathbf{K}_2^{\mathrm{T}}\right), \tag{3.5i}$$

$$\mathbf{K}_1 = (k_{cs} + 2k_s)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.5j}$$

$$\mathbf{K}_2 = k_s\begin{pmatrix} \cos\dfrac{2\pi}{N} & -\sin\dfrac{2\pi}{N} \\ \sin\dfrac{2\pi}{N} & 0 \end{pmatrix}, \tag{3.5k}$$

$$\mathbf{K}_c = \mathrm{diag}(m_c + Nm, m_c + Nm, 0), \tag{3.5l}$$

$$\mathbf{K}_{\mathrm{cs}} = m \begin{pmatrix} \cos\beta_1 & -\sin\beta_1 & \cos\beta_2 & -\sin\beta_2 & \cdots & \cos\beta_N & -\sin\beta_N \\ \sin\beta_1 & \cos\beta_1 & \sin\beta_2 & \cos\beta_2 & \cdots & \sin\beta_N & \cos\beta_N \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{3.5m}$$

and
$$\mathbf{K}_{\mathrm{s}} = \mathrm{diag}\left(\underbrace{m, 0, m, 0, \ldots, m, 0}_{N}\right). \tag{3.5n}$$

The eigenvalue problem of this example conservative system has the form in equation (3.1) with the absence of the damping and circulatory matrices.

(b) Properties of matrix components in equation (3.3)

We require some key properties of the submatrices in equation (3.3), and these are derived first.

(i) Properties of $\mathbf{A}_{\mathrm{s}}$

Imagining the case when all central component translational and rotational vibrations x_p, y_p and μ_p for $p = 1, 2, \ldots, P$ are fixed, even though they are not fixed in the system of interest, the first $3P$ rows and columns in equation (3.3) vanish. Such a system has the structure of a cyclically symmetric system with non-vibrating central components. Therefore, $\mathbf{M}_{\mathrm{s}}$, $\mathbf{G}_{\mathrm{s}}$, $\mathbf{C}_{\mathrm{s}}$, $\mathbf{K}_{\mathrm{s}}$ and $\mathbf{H}_{\mathrm{s}}$ (and so $\mathbf{A}_{\mathrm{s}}$) are block circulant according to the same argument as used earlier. The $\mathbf{A}_{\mathrm{s}}$ (and so the $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N$ that are analogous to the $\mathbf{B}_i$ in equation (2.2)) derived from such a system is identical to the $\mathbf{A}_{\mathrm{s}}$ derived for the same system with central component motions. This block circulant form is evident in the example system, specifically equations (3.5*c*,*f*,*i*,*n*). (Note that diagonal matrices are block circulant.)

(ii) Properties of $\mathbf{A}_{\mathrm{cs}}$

The submatrix operator $\mathbf{A}_{\mathrm{cs}}$ in equation (3.3*a*) connecting the central components with the substructures can be expressed as

$$\mathbf{A}_{\mathrm{cs}} = (\mathbf{A}_{\mathrm{cs}}^{(1)} \quad \mathbf{A}_{\mathrm{cs}}^{(2)} \quad \cdots \quad \mathbf{A}_{\mathrm{cs}}^{(N)}), \tag{3.6}$$

where each $3P \times L$-dimensional $\mathbf{A}_{\mathrm{cs}}^{(i)}$ captures the connection between the ith substructure and the central components. We now show that the $\mathbf{A}_{\mathrm{cs}}^{(i)}$ are related according to

$$\mathbf{A}_{\mathrm{cs}}^{(i)} = \mathrm{diag}\left(\underbrace{\mathbf{R}_i, \mathbf{R}_i, \ldots, \mathbf{R}_i}_{P}\right) \mathbf{A}_{\mathrm{cs}}^{(1)} \tag{3.7a}$$

and

$$\mathbf{R}_i = \begin{pmatrix} \cos\beta_i & -\sin\beta_i & 0 \\ \sin\beta_i & \cos\beta_i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad i = 1, 2, \ldots, N. \tag{3.7b}$$

The submatrices $\mathbf{A}_{\mathrm{cs}}^{(i)}$ are determined by the forces and accelerations associated with the relative motions between the ith substructure and the central components. In a free-body diagram and kinematic analysis, these relative force and acceleration vectors for degrees of freedom in substructure i, when calculated in the basis $\{\mathbf{e}_1^{(i)}, \mathbf{e}_2^{(i)}, \mathbf{e}_3^{(i)}\}$ fixed to the corresponding substructure, have the same components for any i because of cyclic symmetry. When the central component equations of motion are derived in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis, however, these forces and accelerations should be projected onto this reference frame. This projection leads to the form in equation (3.7), where the $\sin\beta_i$ and $\cos\beta_i$ elements in $\mathbf{R}_i$ transform vector components between the $\{\mathbf{e}_1^{(i)}, \mathbf{e}_2^{(i)}, \mathbf{e}_3^{(i)}\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ bases.

We take the absorber acceleration in cyclically symmetric CPVA systems as an example (figure 3). The matrix equations of motion of CPVA systems are in [23]. The tangential acceleration

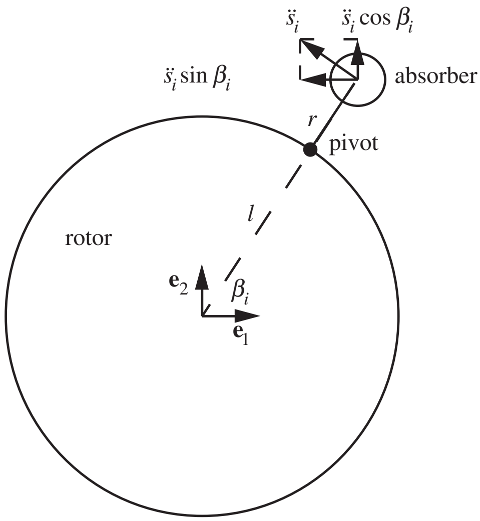

Figure 3. Absorber acceleration vector and components in CPVA systems.

of the ith absorber relative to its pivot point acceleration expressed in the $\{\mathbf{e}_1^{(i)}, \mathbf{e}_2^{(i)}, \mathbf{e}_3^{(i)}\}$ basis is $\ddot{s}_i\mathbf{e}_2^{(i)}$, where s_i is the absorber displacement along its path (figure 3). When calculating the equations of motion that govern the rotor x- and y-translations, this acceleration is projected onto the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis as $-\ddot{s}\sin\beta_i\mathbf{e}_1 + \ddot{s}\cos\beta_i\mathbf{e}_2$. Because of this projection, and after using standard vector methods to derive the governing equations, the inertia submatrix relating the ith absorber and the rotor ($\mathbf{M}_{\text{cs}}^{(i)}$) is determined from the submatrix relating the first absorber and the rotor ($\mathbf{M}_{\text{cs}}^{(1)}$) according to

$$\mathbf{M}_{\text{cs}}^{(i)} = \begin{pmatrix} -m\sin\beta_i \\ m\cos\beta_i \\ m(l+r) \end{pmatrix} = \begin{pmatrix} \cos\beta_i & -\sin\beta_i & 0 \\ \sin\beta_i & \cos\beta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 \\ m \\ m(l+r) \end{pmatrix}}_{=\mathbf{M}_{\text{cs}}^{(1)}}, \quad i = 1, 2, \ldots, N, \tag{3.8}$$

where l and r are the distance from the central axis to the absorber pivot and the absorber radius, respectively. The relation between the submatrices $\mathbf{M}_{\text{cs}}^{(1)}$ and $\mathbf{M}_{\text{cs}}^{(i)}$ in equation (3.8) is exactly the same as that expressed in equation (3.7). This property can also be observed in the gyroscopic and stiffness matrices of cyclically symmetric CPVA systems [23–25], where the submatrices are related according to

$$\mathbf{G}_{\text{cs}}^{(i)} = \begin{pmatrix} -m\cos\beta_i \\ -m\sin\beta_i \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\beta_i & -\sin\beta_i & 0 \\ \sin\beta_i & \cos\beta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -m \\ 0 \\ 0 \end{pmatrix}}_{=\mathbf{G}_{\text{cs}}^{(1)}} \tag{3.9a}$$

and

$$\mathbf{K}_{\text{cs}}^{(i)} = \begin{pmatrix} -m\sin\beta_i \\ m\cos\beta_i \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\beta_i & -\sin\beta_i & 0 \\ \sin\beta_i & \cos\beta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 \\ m \\ 0 \end{pmatrix}}_{=\mathbf{K}_{\text{cs}}^{(1)}}, \quad i = 1, 2, \ldots, N. \tag{3.9b}$$

The $3 \times N$ (for the CPVA system) matrix $\mathbf{A}_{\text{cs}}$ in equation (3.6) is assembled from equations (3.8) and (3.9), so it has the structure shown in equation (3.7).

Planetary gears with matrices given in [19] provide another example. Considering the stiffness matrix, the submatrix relating the ith planet (substructure) and the sun gear (central component) is, using the notation in [19],

$$\mathbf{K}_{\mathrm{cs}}^{(i)} = k_{sp}\begin{pmatrix} \sin(\beta_i-\alpha_s)\sin\alpha_s & \sin(\beta_i-\alpha_s)\cos\alpha_s & -\sin(\beta_i-\alpha_s) \\ -\cos(\beta_i-\alpha_s)\sin\alpha_s & -\cos(\beta_i-\alpha_s)\cos\alpha_s & \cos(\beta_i-\alpha_s) \\ -\sin\alpha_s & -\cos\alpha_s & 1 \end{pmatrix}$$

$$= k_{sp}\begin{pmatrix} \cos\beta_i & -\sin\beta_i & 0 \\ \sin\beta_i & \cos\beta_i & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\sin^2\alpha_s & -\sin\alpha_s\cos\alpha_s & \sin\alpha_s \\ -\sin\alpha_s\cos\alpha_s & -\cos^2\alpha_s & \cos\alpha_s \\ -\sin\alpha_s & -\cos\alpha_s & 1 \end{pmatrix}, \tag{3.10}$$

where k_{sp} and α_s are system parameters and the last matrix in equation (3.10) is $\mathbf{K}_{\mathrm{cs}}^{(1)}$. Note that $\mathbf{K}_{\mathrm{cs}}^{(1)} = \mathbf{K}_{\mathrm{cs}}^{(i)}$ for $\beta_i = 0$, as expected, which also holds for equations (3.8) and (3.9). The submatrices that relate the ith planet with the ring gear and carrier (central components) can be decomposed in the same way. The mass and gyroscopic matrices of planetary gears are block-diagonal, so no submatrices $\mathbf{M}_{\mathrm{cs}}$ or $\mathbf{G}_{\mathrm{cs}}$ exist. Thus, for planetary gears where $P = 3$ and $L = 3$, the 9×3 matrix $\mathbf{A}_{\mathrm{cs}}^{(i)}$ obtained from [19] has the structure of equation (3.7).

The matrix $\mathbf{A}_{\mathrm{cs}}$ for the example system in figure 2 also satisfies equation (3.7). The submatrices of $\mathbf{M}_{\mathrm{cs}}$, $\mathbf{G}_{\mathrm{cs}}$ and $\mathbf{K}_{\mathrm{cs}}$ relating the central component and substructure motions are

$$\mathbf{M}_{\mathrm{cs}}^{(i)} = m\begin{pmatrix} \cos\beta_i & -\sin\beta_i \\ \sin\beta_i & \cos\beta_i \\ 0 & l \end{pmatrix} = m\begin{pmatrix} \cos\beta_i & -\sin\beta_i & 0 \\ \sin\beta_i & \cos\beta_i & 0 \\ 0 & 0 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & l \end{pmatrix}}_{\mathbf{M}_{\mathrm{cs}}^{(1)}}, \tag{3.11a}$$

$$\mathbf{G}_{\mathrm{cs}}^{(i)} = m\begin{pmatrix} -\sin\beta_i & -\cos\beta_i \\ \cos\beta_i & -\sin\beta_i \\ -l & 0 \end{pmatrix} = m\begin{pmatrix} \cos\beta_i & -\sin\beta_i & 0 \\ \sin\beta_i & \cos\beta_i & 0 \\ 0 & 0 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -l & 0 \end{pmatrix}}_{\mathbf{G}_{\mathrm{cs}}^{(1)}} \tag{3.11b}$$

and $$\mathbf{K}_{\mathrm{cs}}^{(i)} = m\begin{pmatrix} \cos\beta_i & -\sin\beta_i \\ \sin\beta_i & \cos\beta_i \\ 0 & 0 \end{pmatrix} = m\begin{pmatrix} \cos\beta_i & -\sin\beta_i & 0 \\ \sin\beta_i & \cos\beta_i & 0 \\ 0 & 0 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\mathbf{K}_{\mathrm{cs}}^{(i)}}, \quad i = 1, 2, \ldots, N. \tag{3.11c}$$

The 3×2 submatrices of the $3 \times 2N$ $\mathbf{A}_{\mathrm{cs}}$ in equation (3.6) are

$$\mathbf{A}_{\mathrm{cs}}^{(i)} = \lambda^2\mathbf{M}_{\mathrm{cs}}^{(i)} + 2\Omega\lambda\mathbf{G}_{\mathrm{cs}}^{(i)} - \Omega^2\mathbf{K}_{\mathrm{cs}}^{(i)}, \quad i = 1, 2, \ldots, N. \tag{3.12}$$

A second explanation of equation (3.7) is based on the matrix $\mathbf{A}_{\mathrm{cs}}^{(i)}$ being the components of a second-order tensor $\tilde{\mathbf{A}}_{\mathrm{cs}}^{(i)}$ (where the tilde distinguishes the tensor from its matrix of components). To show this more easily, we consider below the case of a single central component ($P = 1$) with 3 degrees of freedom and only 3 degrees of freedom for each substructure ($L = 3$), which gives tensor algebra in terms of 3×3 matrix components. We also refer to tensors and matrices associated with $\mathbf{A}_{\mathrm{cs}}$, although the steps below can be applied individually to the inertia, gyroscopic, damping, stiffness and circulatory components of $\mathbf{A}_{\mathrm{cs}} = \lambda^2\mathbf{M}_{\mathrm{cs}} + \lambda(\mathbf{G}_{\mathrm{cs}} + \mathbf{C}_{\mathrm{cs}}) + \mathbf{K}_{\mathrm{cs}} + \mathbf{H}_{\mathrm{cs}}$ and summed to form $\mathbf{A}_{\mathrm{cs}}$.

The matrix $\mathbf{A}_{\mathrm{cs}}^{(1)}$, which is 3×3 for $P = 1$ and $L = 3$, contains the components of the tensor $\tilde{\mathbf{A}}_{\mathrm{cs}}^{(1)}$ calculated in the $\{\mathbf{e}_m^{(1)} \otimes \mathbf{e}_n^{(1)}\}$ basis, which is identical to the $\{\mathbf{e}_m \otimes \mathbf{e}_n^{(1)}\}$ basis because $\beta_1 = 0$. (The second-order tensor $\mathbf{a} \otimes \mathbf{b}$ is such that $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$ for vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.) Mathematically, this is stated as

$$\tilde{\mathbf{A}}_{\mathrm{cs}}^{(1)} = \sum_{m=1}^{3}\sum_{n=1}^{3}(\mathbf{A}_{\mathrm{cs}}^{(1)})_{mn}\mathbf{e}_m \otimes \mathbf{e}_n^{(1)} = \mathbf{A}_{\mathrm{cs}}^{(1)}\mathbf{e}_m \otimes \mathbf{e}_n^{(1)}, \tag{3.13}$$

where, for notational simplicity in the second equality, we omit the subscript mn denoting the (m, n) components of the matrix $\mathbf{A}_{\mathrm{cs}}^{(1)}$ and the repeated indices m and n imply summation over the range 1, 2 and 3. Because of cyclic symmetry, the components of the tensor $\tilde{\mathbf{A}}_{\mathrm{cs}}^{(i)}$ calculated in the basis $\{\mathbf{e}_m^{(i)} \otimes \mathbf{e}_n^{(i)}\}$ are identical to the components of $\tilde{\mathbf{A}}_{\mathrm{cs}}^{(1)}$ calculated in the $\{\mathbf{e}_m^{(1)} \otimes \mathbf{e}_n^{(1)}\}$ basis. Thus, $\tilde{\mathbf{A}}_{\mathrm{cs}}^{(i)} = \mathbf{A}_{\mathrm{cs}}^{(1)} \mathbf{e}_m^{(i)} \otimes \mathbf{e}_n^{(i)}$. For central component equations of motion in terms of coordinates in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis and substructure degrees of freedom defined in the $\{\mathbf{e}_1^{(i)}, \mathbf{e}_2^{(i)}, \mathbf{e}_3^{(i)}\}$ bases, however, the matrices $\mathbf{A}_{\mathrm{cs}}^{(i)}$ in equations (3.3*a*) and (3.6) consist of the components of the corresponding tensors $\tilde{\mathbf{A}}_{\mathrm{cs}}^{(i)}$ on the mixed basis $\{\mathbf{e}_m \otimes \mathbf{e}_n^{(i)}\}$.

To perform the tensor algebra to find the matrix $\mathbf{A}_{\mathrm{cs}}^{(i)}$ consisting of the components of $\tilde{\mathbf{A}}_{\mathrm{cs}}^{(i)}$ on the mixed basis $\{\mathbf{e}_m \otimes \mathbf{e}_n^{(i)}\}$ starting from the known matrix $\mathbf{A}_{\mathrm{cs}}^{(1)}$ of components of $\tilde{\mathbf{A}}_{\mathrm{cs}}^{(i)}$ on the basis $\{\mathbf{e}_m^{(i)} \otimes \mathbf{e}_n^{(i)}\}$, we first relate the two sets of base vectors using the proper orthogonal rotation tensor $\tilde{\mathbf{R}}$ as

$$\mathbf{e}_m^{(i)} = \tilde{\mathbf{R}}_i \mathbf{e}_m \tag{3.14a}$$

and

$$\tilde{\mathbf{R}}_i = \mathbf{R}_i \mathbf{e}_m \otimes \mathbf{e}_n = \mathbf{R}_i \mathbf{e}_m^{(i)} \otimes \mathbf{e}_n^{(i)}, \tag{3.14b}$$

where the matrix $\mathbf{R}_i$ is given in equation (3.7*b*). Note the matrix components of $\tilde{\mathbf{R}}_i$ are the same on both the $\{\mathbf{e}_m^{(i)} \otimes \mathbf{e}_n^{(i)}\}$ and $\{\mathbf{e}_m \otimes \mathbf{e}_n\}$ bases. By standard tensor algebra, the components of $\tilde{\mathbf{A}}_{\mathrm{cs}}^{(i)}$ on the mixed basis $\{\mathbf{e}_m \otimes \mathbf{e}_n^{(i)}\}$ satisfy

$$\tilde{\mathbf{A}}_{\mathrm{cs}}^{(i)} = \mathbf{A}_{\mathrm{cs}}^{(1)} \mathbf{e}_m^{(i)} \otimes \mathbf{e}_n^{(i)} = \mathbf{A}_{\mathrm{cs}}^{(i)} \mathbf{e}_m \otimes \mathbf{e}_n^{(i)} \tag{3.15a}$$

and

$$\mathbf{A}_{\mathrm{cs}}^{(i)} = \mathbf{R}_i \mathbf{A}_{\mathrm{cs}}^{(1)}. \tag{3.15b}$$

The matrix (not tensor) relationship $\mathbf{A}_{\mathrm{cs}}^{(i)} = \mathbf{R}_i \mathbf{A}_{\mathrm{cs}}^{(1)}$ in equation (3.15*b*) is exactly that in equation (3.7). From this special case of $P = 1$ and $L = 3$, one can generalize the above argument to confirm that equation (3.7) holds for arbitrary P and L.

(iii) Properties of $\mathbf{A}_{\mathrm{c}}$

The submatrix $\mathbf{A}_{\mathrm{c}}$ in equation (3.3*a*) associated only with the central component motions can be written as

$$\mathbf{A}_{\mathrm{c}} = \begin{pmatrix} \mathbf{A}_{11}^{\mathrm{c}} & \mathbf{A}_{12}^{\mathrm{c}} & \cdots & \mathbf{A}_{1P}^{\mathrm{c}} \\ \mathbf{A}_{21}^{\mathrm{c}} & \mathbf{A}_{22}^{\mathrm{c}} & \cdots & \mathbf{A}_{2P}^{\mathrm{c}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{P1}^{\mathrm{c}} & \mathbf{A}_{P2}^{\mathrm{c}} & \cdots & \mathbf{A}_{PP}^{\mathrm{c}} \end{pmatrix}, \tag{3.16}$$

where $\mathbf{A}_{pq}^{\mathrm{c}}$ for $p, q = 1, 2, \ldots, P$ are 3×3 submatrices that couple the motions of the pth and qth central components. The components $a_{11}^{(pq)}$, $a_{12}^{(pq)}$, $a_{21}^{(pq)}$ and $a_{22}^{(pq)}$ of $\mathbf{A}_{pq}^{\mathrm{c}}$ in equation (3.16) relating the central component translations satisfy $a_{11}^{(pq)} = a_{22}^{(pq)}$ and $a_{12}^{(pq)} = -a_{21}^{(pq)}$. These properties can be proved for all cyclically symmetric systems, including those with coupled central component translations and rotations. For convenience, however, the matrix elements that couple the central component translations and rotations are not considered in the following proof because we later assume these elements vanish, i.e. $a_{13}^{(pq)} = a_{23}^{(pq)} = a_{31}^{(pq)} = a_{32}^{(pq)} = 0$. For concreteness, and without loss of generality, the degree-of-freedom definition below equation (3.2) stipulates that x and y central component displacements are along the $\mathbf{e}_1$ and $\mathbf{e}_2$ base vectors. Because the central components and any stiffness connections between two central components are axisymmetric, however, the elements of the submatrices $\mathbf{A}_{pq}^{\mathrm{c}}$ (but not necessarily other elements of the system matrices in equation (3.3)) must be unchanged if x_p, y_p, x_q and y_q are along two other arbitrarily chosen orthogonal directions in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ plane. In other words, the submatrix $\mathbf{A}^{c}_{pq}$ is invariant under the rotation of an arbitrary angle θ, that is,

$$\mathbf{A}^{c}_{pq} = \mathbf{R}^{\mathrm{T}}\mathbf{A}^{c}_{pq}\mathbf{R}, \quad \mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.17}$$

This gives

$$\begin{pmatrix} a^{(pq)}_{11} & a^{(pq)}_{12} & 0 \\ a^{(pq)}_{21} & a^{(pq)}_{22} & 0 \\ 0 & 0 & a^{(pq)}_{33} \end{pmatrix}$$
$$= \begin{pmatrix} a^{(pq)}_{11}\cos^2\theta + a^{(pq)}_{22}\sin^2\theta + (a^{(pq)}_{12} + a^{(pq)}_{21})\sin\theta\cos\theta & a^{(pq)}_{12}\cos^2\theta - a^{(pq)}_{21}\sin^2\theta - (a^{(pq)}_{11} - a^{(pq)}_{22})\sin\theta\cos\theta & 0 \\ a^{(pq)}_{21}\cos^2\theta - a^{(pq)}_{12}\sin^2\theta - (a^{(pq)}_{11} - a^{(pq)}_{22})\sin\theta\cos\theta & a^{(pq)}_{11}\sin^2\theta + a^{(pq)}_{22}\cos^2\theta - (a^{(pq)}_{12} + a^{(pq)}_{21})\sin\theta\cos\theta & 0 \\ 0 & 0 & a^{(pq)}_{33} \end{pmatrix}. \tag{3.18}$$

Equation (3.18) must hold for arbitrary θ. This requires $a^{(pq)}_{11} = a^{(pq)}_{22}$ (e.g. substitute $\theta = \pi/2$) and $a^{(pq)}_{12} = -a^{(pq)}_{21}$ (e.g. substitute $\theta = \pi/4$) as the necessary and sufficient conditions. With these properties, each submatrix has the form

$$\mathbf{A}^{c}_{pq} = \begin{pmatrix} a^{(pq)}_{11} & a^{(pq)}_{12} & 0 \\ a^{(pq)}_{21} & a^{(pq)}_{22} & 0 \\ 0 & 0 & a^{(pq)}_{33} \end{pmatrix} = \begin{pmatrix} a^{(pq)}_{11} & -a^{(pq)}_{21} & 0 \\ a^{(pq)}_{21} & a^{(pq)}_{11} & 0 \\ 0 & 0 & a^{(pq)}_{33} \end{pmatrix}. \tag{3.19}$$

For general cyclically symmetric systems with coupled central component translations and rotations, one can get the same result using a similar derivation.

(c) Simplified modelling based on cyclic symmetry

The modelling of dynamic systems, whether by directly deriving the equations of motion or by finite elements, usually requires study of the entire system. The modelling of cyclically symmetric systems, however, can be simplified substantially by applying the above properties of the matrices in equation (3.3). Later, we will show how these matrix properties also dramatically reduce the size of the eigenvalue problems to be solved, which is potentially a great computational benefit for large numbers of substructures or for many degrees of freedom per substructure.

When modelling the submatrix $\mathbf{A}_{\mathrm{s}}$ in equation (3.3*a*) associated only with the substructure motions, the central components and all substructure connections to the central components play no part. We can neglect them when deriving $\mathbf{A}_{\mathrm{s}}$. Because $\mathbf{A}_{\mathrm{s}}$ is block circulant with the form shown in equation (2.2), it can be formulated entirely from the $\mathbf{A}_i$ for $i = 1, 2, \ldots, N$ in equation (2.3). Thus, we only need $\mathbf{A}^{(1)}_{\mathrm{s}}$ associated with the first substructure. $\mathbf{A}^{(1)}_{\mathrm{s}}$, and all of its submatrices $\mathbf{A}_i$, are calculated by considering only the first substructure and any substructures connected to the first substructure. $\mathbf{A}_i = \mathbf{0}$ for substructures that are not connected to the first substructure. For the example in figure 2*a*, one can use the simplified model shown in figure 2*b* that contains only the first, the second and the Nth substructures, and no central component, to obtain $\mathbf{A}^{(1)}_{\mathrm{s}}$ and therefore $\mathbf{A}_{\mathrm{s}}$.

The submatrix $\mathbf{A}_{\mathrm{cs}}$ coupling the substructure motions and central component motions consists of the submatrices $\mathbf{A}^{(i)}_{\mathrm{cs}}$ for $i = 1, 2, \ldots, N$ in equation (3.6). Based on equation (3.7), all of the submatrices $\mathbf{A}^{(i)}_{\mathrm{cs}}$ can be calculated from $\mathbf{A}^{(1)}_{\mathrm{cs}}$ that couples the first substructure with the central components. Therefore, $\mathbf{A}_{\mathrm{cs}}$ can be formulated entirely by studying only the first substructure and the central components. For the example system, we only need to consider the simplified model in figure 2*c* to formulate $\mathbf{A}_{\mathrm{cs}}$. The submatrix $\mathbf{A}_{\mathrm{sc}}$ in equation (3.3*a*) is obtained by applying the symmetric and skew-symmetric properties of $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ and $\mathbf{G}$, $\mathbf{H}$ to the calculated submatrix $\mathbf{A}_{\mathrm{cs}}$.

The submatrix $\mathbf{A}_c$ in equation (3.3*a*) involves only the central component motions, so only the central components are needed in its formulation; the N substructures can be neglected. Thus, $\mathbf{A}_c$ of the example system can be calculated using the simplified model in figure 2*d* that contains only the central components, just one in this example.

According to the above discussion, only the central components, the first substructure and the substructures that are connected to the first substructure are needed in the modelling of general cyclically symmetric systems. All other matrices can be derived from the matrices obtained from the above subsystem models. Working with these simpler subsystem models reduces effort and errors for cases like planetary gears, CPVAs and the example in figure 2. Modelling effort and computational expense are reduced dramatically when using the simplified subsystem models with large-scale finite-element analysis. Note the full system matrices in equation (3.3) generated from these subsystem model matrices can be used for dynamic response and control purposes, not only eigenvalue problem analysis.

(d) Modal decomposition

We previously partitioned the eigenvectors for cyclically symmetric systems with central components into central component ($\boldsymbol{\phi}_c$) and substructure ($\boldsymbol{\phi}_s$) motions according to equation (3.2). Now, we stipulate, and subsequently prove, that the substructure degrees of freedom have the form

$$\boldsymbol{\phi}_s = (e^{jk\beta_1}\tilde{\boldsymbol{\phi}}_s, e^{jk\beta_2}\tilde{\boldsymbol{\phi}}_s, \ldots, e^{jk\beta_N}\tilde{\boldsymbol{\phi}}_s)^T, \quad k = 0, 1, \ldots, N-1, \tag{3.20}$$

where $\tilde{\boldsymbol{\phi}}_s$ is L-dimensional. The vibration modes are later categorized into substructure and coupled modes based on whether $\boldsymbol{\phi}_c$ vanishes or not. The coupled modes can be further categorized into translational and rotational modes for systems with uncoupled central component translations and rotations. We will derive the properties of $\boldsymbol{\phi}_c$ that motivate these names. We will also show that $\boldsymbol{\phi}_s$ has the form in equation (3.20) for every mode type. The integer $k \in \{0, 1, 2, \ldots, N-1\}$ in equation (3.20) differs for each mode type.

Examination of equation (3.20) reveals that the motion of every substructure is the same except for a phase difference determined by the phase index k and spacing angle β_i. This also means the modal amplitudes (magnitudes) of corresponding degrees of freedom in every substructure are the same. Because $\tilde{\boldsymbol{\phi}}_s$ is complex, in general, the degrees of freedom within a given substructure are not in phase.

The eigenvectors in equation (3.2) must satisfy the system eigenvalue problem in equation (3.3). Substitution of equation (3.2) into the equations that govern the central component motions in equation (3.3) yields

$$\mathbf{A}_c\boldsymbol{\phi}_c + \mathbf{A}_{cs}\boldsymbol{\phi}_s = \mathbf{0}. \tag{3.21}$$

Invoking equations (3.20) and (3.7), the second term of equation (3.21) is

$$\mathbf{A}_{cs}\boldsymbol{\phi}_s = \operatorname{diag}\left(\underbrace{\hat{\mathbf{R}}, \hat{\mathbf{R}}, \ldots, \hat{\mathbf{R}}}_{P}\right)\mathbf{A}_{cs}^{(1)}\tilde{\boldsymbol{\phi}}_s \tag{3.22a}$$

and

$$\hat{\mathbf{R}} = \begin{pmatrix} \sum_{i=1}^{N} e^{jk\beta_i}\cos\beta_i & -\sum_{i=1}^{N} e^{jk\beta_i}\sin\beta_i & 0 \\ \sum_{i=1}^{N} e^{jk\beta_i}\sin\beta_i & \sum_{i=1}^{N} e^{jk\beta_i}\cos\beta_i & 0 \\ 0 & 0 & \sum_{i=1}^{N} e^{jk\beta_i} \end{pmatrix}. \tag{3.22b}$$

(i) Substructure modes

We first consider the cases $k=2,3,\ldots,N-2$. For such k, $\hat{\mathbf{R}}$ in equation (3.22) vanishes because of the trigonometric identities $\sum_{i=1}^{N} \mathrm{e}^{jk\beta_i}\cos\beta_i = \sum_{i=1}^{N} \mathrm{e}^{jk\beta_i}\sin\beta_i = \sum_{i=1}^{N} \mathrm{e}^{jk\beta_I} = 0$ for $k=2,3,\ldots,N-2$ [23]. Therefore, the second term in equation (3.21) vanishes for $k=2,3,\ldots,N-2$. In planetary gears [6,19–22,34] and CPVA systems [23–25], the vibration modes with phase index $k=2,3,\ldots,N-2$ are planet/absorber modes, where no central component motions exist. When generalizing to an arbitrary cyclically symmetric system, we propose that modes with $k=2,3,\ldots,N-2$ have zero central component motions $\boldsymbol{\phi}_{\mathrm{c}}=\mathbf{0}$. Thus, equation (3.21) is satisfied for $k=2,3,\ldots,N-2$ and $\boldsymbol{\phi}_{\mathrm{c}}=\mathbf{0}$.

It remains only to satisfy the substructure equations in equation (3.3). With $\boldsymbol{\phi}_{\mathrm{c}}=\mathbf{0}$, substitution of the eigenvectors in equations (3.2) and (3.20) into the ith substructure equations of the eigenvalue problem in equation (3.3) and multiplication by $\mathrm{e}^{-jk\beta_i}$ for $i=1,2,\ldots,N$ all yield (for any i)

$$\left(\sum_{i=1}^{N} \mathrm{e}^{jk\beta_i}\mathbf{A}_i\right)\tilde{\boldsymbol{\phi}}_{\mathrm{s}}=\mathbf{0},\quad k=2,3\ldots,N-2, \tag{3.23}$$

where the $L\times L$ matrices $\mathbf{A}_i$ are the submatrices of the block circulant matrix $\mathbf{A}_{\mathrm{s}}$ given in equation (3.3). Equation (3.23) is identical to equation (2.6). Thus, the L eigenvectors $\tilde{\boldsymbol{\phi}}_{\mathrm{s}}$ of the $L\times L$ reduced eigenvalue problem in equation (3.23) are the same as the L reduced eigenvectors $\mathbf{v}_{kl}$ given in equation (2.6) for $k=2,3,\ldots,N-2$. Therefore, the $\boldsymbol{\phi}_{\mathrm{s}}$ for $k=2,3,\ldots,N-2$ and $\tilde{\boldsymbol{\phi}}_{\mathrm{s}}$ given by equation (3.23) are the same as the $\mathbf{u}_{kl}$ in equation (2.5) with the $\mathbf{v}_{kl}$ from equation (2.6); the modes with $k=2,3,\ldots,N-2$ are identical to those for the system with non-vibrating central components. The eigenvalues are also identical. The calculated $\tilde{\boldsymbol{\phi}}_{\mathrm{s}}$ are substituted, along with $\boldsymbol{\phi}_{\mathrm{c}}=\mathbf{0}$, into equation (3.20) to give the full system eigenvectors for $k=2,3,\ldots,N-2$

$$\boldsymbol{\phi}=(\mathbf{0}_{1\times 3P},\mathrm{e}^{jk\beta_1}\tilde{\boldsymbol{\phi}}_{\mathrm{s}},\mathrm{e}^{jk\beta_2}\tilde{\boldsymbol{\phi}}_{\mathrm{s}},\ldots,\mathrm{e}^{jk\beta_N}\tilde{\boldsymbol{\phi}}_{\mathrm{s}})^{\mathrm{T}}. \tag{3.24}$$

These $(N-3)L$ eigenvectors having no central component motion for $k=2,3,\ldots,N-2$ are called *substructure modes* because the modal deformation is solely in the substructures.

The substructure modes and their eigenvalues are determined fully from the $N-3$ $L\times L$ eigenvalue problems in equation (3.23). One does not need to formulate or solve the much bigger $(NL+3P)\times(NL+3P)$ full system eigenvalue problem in equation (3.3).

Substructure modes associated with $k=z$ and $k=N-z$, $z=2,3,\ldots,N-2$ and $z\neq N/2$ are complex conjugate of each other. For the case when there are an even number of substructures, the substructure modes in equation (3.24) associated with $k=N/2$ are real-valued because $\beta_i\cdot N/2=[2\pi(i-1)/N]\cdot(N/2)=\pi(i-1)$ and $\mathrm{e}^{[j(i-1)\pi]}$ is real for $i=1,2,\ldots,N$.

In cases like planetary gears [6,19–22,34] and CPVA systems [23–25] where the substructures do not connect with each other but only connect to the central components, the submatrix $\mathbf{A}_{\mathrm{s}}$ is block-diagonal. The $\mathbf{A}_i$ (defined in equations (2.3) and (2.4)) vanish for $i\neq 1$. In this case, for any $k=2,3,\ldots,N-2$ equation (3.23) yields the same L-dimensional eigenvalue problem

$$\mathbf{A}_1\tilde{\boldsymbol{\phi}}_{\mathrm{s}}=\mathbf{0},\quad k=2,3,\ldots,N-2. \tag{3.25}$$

Equation (3.25) gives the same L eigenvalues and eigenvectors $\tilde{\boldsymbol{\phi}}_{\mathrm{s}}$ for any $k\in\{2,3,\ldots,N-2\}$. Thus, each of the L substructure mode eigenvalues for the complete cyclically symmetric system (governed by equation (3.3)) is degenerate with multiplicity $N-3$. While the $\tilde{\boldsymbol{\phi}}_{\mathrm{s}}$ from equation (3.25) are the same for each $k\in\{2,3,\ldots,N-2\}$, the $\boldsymbol{\phi}$ in equation (3.20) differ for every k. Thus, for each degenerate eigenvalue λ, we obtain $N-3$ independent eigenvectors $\boldsymbol{\phi}$ of the form in equation (3.20) with $\boldsymbol{\phi}_{\mathrm{c}}=\mathbf{0}$. Because we know there are $(N-3)L$ substructure modes for a general cyclically symmetric system as determined above, we conclude there are L different degenerate eigenvalues with each having multiplicity $N-3$. These results generalize to arbitrary cyclically symmetric systems the modal properties derived specifically for planetary gear [6,19–22,34] and CPVA [23–25] systems. The substructure modes were referred to as *planet* and *absorber modes* in those prior works. As a reminder, the results in this paragraph are for the special case where the substructures do not connect to each other; the general case of substructure modes requires equation (2.6), where degeneracy of the $(N-3)L$ substructure mode eigenvalues is not expected.

(ii) Coupled modes

For $k=0,1$ and $N-1$, the second term in equation (3.21) does not vanish, as it did for $k=2,3,\ldots,N-2$ in the substructure modes. This requires non-zero $\boldsymbol{\phi}_{\mathrm{c}}$ to satisfy the equation. Because the substructure motions are coupled with non-zero central component motions for $k=0,1$ and $N-1$, these modes are called *coupled modes*. Because the system has $3P+NL$ modes in total and $(N-3)L$ substructure modes, there are $3(P+L)$ coupled modes. Generally, the central component translations and rotations are coupled for coupled modes. For the extremely common case where the central component translations and rotations are uncoupled in the equations of motion as discussed below, the coupled modes can be further categorized into translational and rotational modes (with no other possible mode types) as shown subsequently.

The elements $a_{13}^{(pq)}, a_{23}^{(pq)}, a_{31}^{(pq)}$ and $a_{32}^{(pq)}$ of the $\mathbf{A}_{pq}^{\mathrm{c}}$ in equation (3.16) couple the central component translations and rotations. We are interested in the common case where these elements vanish. Because one can always postulate a stiffness or damping device where central component translation induces a torque on the central component (or central component rotation induces a force), it is not possible to prove the absence of coupling between translation and rotation. Such devices are rare, however, so the vast majority of systems will have no such coupling in the stiffness, circulatory and damping matrices. For the mass and gyroscopic matrices, we can use Lagrange's equations to prove the absence of coupling between central component translation and rotation. The mass and gyroscopic matrices are determined solely by the system kinetic energy, and the current question requires only the total central component kinetic energy. The position vector of the pth central component is

$$\mathbf{r}_p = x_p\mathbf{e}_1 + y_p\mathbf{e}_2. \tag{3.26}$$

The velocity vector for constant rotation speed Ω of the $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ basis is

$$\dot{\mathbf{r}}_p = (\dot{x}_p - \Omega y_p)\mathbf{e}_1 + (\dot{y}_p + \Omega x_p)\mathbf{e}_2. \tag{3.27}$$

The angular speed of the pth central component is

$$\omega_p = \rho_p\Omega + \dot{\mu}_p, \tag{3.28}$$

where ρ_p is the speed ratio between the pth central component and the $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ basis and μ_p is the rotational vibration of the pth central component. The total kinetic energy of the central components is

$$T_{\mathrm{c}} = \sum_{p=1}^{P}\frac{1}{2}\left\{m_p\left[(\dot{x}_p-\Omega y_p)^2 + (\dot{y}_p+\Omega x_p)^2\right] + J_p(\rho_p\Omega+\dot{\mu}_p)^2\right\}, \tag{3.29}$$

where m_p and J_p are the mass and moment of inertia about the central axis of the pth central component. Neither of the partial derivatives $\partial T_{\mathrm{c}}/\partial\dot{x}_p$ and $\partial T_{\mathrm{c}}/\partial x_p$ in Lagrange's equations is a function of μ_p. Likewise, $\partial T_{\mathrm{c}}/\partial\dot{\mu}_p$ and $\partial T_{\mathrm{c}}/\partial\mu_p$ are independent of x_p. Thus, the x-translation and rotation do not couple through the central component kinetic energy, and, similarly, neither do the y-translation and rotation. Therefore, the mass and gyroscopic matrices, which derive entirely from the kinetic energy, do not contribute to the $a_{13}^{(pq)}$, $a_{23}^{(pq)}$, $a_{31}^{(pq)}$ and $a_{32}^{(pq)}$ elements of $\mathbf{A}_{pq}^{\mathrm{c}}$. Hence, when these same four coupling elements in each of the stiffness, damping and circulatory matrices also vanish, as discussed above, the central component translations and rotations are fully uncoupled and $\mathbf{A}_{pq}^{\mathrm{c}}$ has the form in equation (3.19). Most mechanical systems satisfy this condition.

In the following, we show that all coupled modes of general cyclically symmetric systems with uncoupled central component translations and rotations can be categorized as one of translational or rotational modes.

Translational modes. The phase indices $k=1$ and $N-1$ are associated with the translational modes in planetary gears [6,19–22] and CPVA systems [23–25]. In such modes, only central component translations, but no rotations, exist. Guided by these examples in generalizing to arbitrary cyclically symmetric systems, we propose a candidate eigenvector with no central component rotations for $k=1$ and $k=N-1$, that is, $\mu_p=0$ for all $p=1,2,\ldots,P$. Furthermore, and again motivated by planetary gear and CPVA systems, the proposed motion of each central component is $\boldsymbol{\phi}_p^{(t)}=(x_p, jx_p, 0)^{\mathrm{T}}$ for $p=1,2,\ldots,P$, where the superscript (t) denotes a translational mode. It remains to show that eigenvectors with these properties actually satisfy the eigenvalue problem. For convenience, we consider the case when $k=1$ first. The case when $k=N-1$ is the complex conjugate of that when $k=1$.

We consider equation (3.21) as P sets of equations for $p=1,2,\ldots,P$ with three equations (rows) per set. Substitution of equation (3.2) with the central component motions $\boldsymbol{\phi}_p^{(t)}=(x_p, jx_p, 0)^{\mathrm{T}}$ into the first term of equation (3.21) and use of equations (3.16) and (3.19) for each $p=1,2,\ldots,P$ yield

$$\sum_{q=1}^{P}\mathbf{A}^{\mathrm{c}}_{pq}\boldsymbol{\phi}_q=\sum_{q=1}^{P}\begin{pmatrix} a_{11}^{(pq)} & -a_{21}^{(pq)} & 0\\ a_{21}^{(pq)} & a_{11}^{(pq)} & 0\\ 0 & 0 & a_{33}^{(pq)}\end{pmatrix}\begin{pmatrix} x_q\\ jx_q\\ 0\end{pmatrix}=\sum_{q=1}^{P}\begin{pmatrix}(a_{11}^{(pq)}-ja_{21}^{(pq)})x_q\\ (a_{21}^{(pq)}+ja_{11}^{(pq)})x_q\\ 0\end{pmatrix},\quad p=1,2,\ldots,P. \tag{3.30}$$

To consider the second term of equation (3.21), the $\hat{\mathbf{R}}$ in equation (3.22) for $k=1,N-1$ reduces to

$$\hat{\mathbf{R}}=\begin{cases}\begin{pmatrix}\dfrac{N}{2} & -j\dfrac{N}{2} & 0\\ j\dfrac{N}{2} & \dfrac{N}{2} & 0\\ 0 & 0 & 0\end{pmatrix}; & k=1\\[2ex] \begin{pmatrix}\dfrac{N}{2} & j\dfrac{N}{2} & 0\\ -j\dfrac{N}{2} & \dfrac{N}{2} & 0\\ 0 & 0 & 0\end{pmatrix}; & k=N-1.\end{cases} \tag{3.31}$$

For every $p=1,2,\ldots,P$, the third element of equation (3.30) vanishes, as does the corresponding row of the second term in equation (3.21) based on the $\hat{\mathbf{R}}$ in equation (3.31). Thus, the P equations governing central component rotations vanish. Comparison of the first two rows of equation (3.30) and the first two rows generated by the second term in equation (3.21) (from equation (3.22*a*) and the $\hat{\mathbf{R}}$ in equation (3.31)) shows that the equation governing the y-translation of each central component is j times the x-translation equation. Only one of the two equations is independent for each central component. Therefore, the $3P$ equations for the central component motions generate only P independent equations from the central component equations.

Substitution of the assumed eigenvector for $k=1$ into the equations that govern the ith substructure motions in equation (3.3), use of the relation between $\mathbf{A}_{\mathrm{cs}}^{(i)}$ and $\mathbf{A}_{\mathrm{cs}}^{(1)}$ derived in equation (3.7), and use of the symmetry of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ and the skew-symmetry of $\mathbf{G}$ and $\mathbf{H}$ yield (see appendix A)

$$\mathbf{A}_{\mathrm{sc}}^{(1)}\mathrm{diag}\Big(\underbrace{\mathbf{R}_i^{\mathrm{T}},\mathbf{R}_i^{\mathrm{T}},\ldots,\mathbf{R}_i^{\mathrm{T}}}_{P}\Big)(\boldsymbol{\phi}_1^{(t)},\boldsymbol{\phi}_2^{(t)},\ldots,\boldsymbol{\phi}_P^{(t)})^{\mathrm{T}}+\left(\sum_{q=1}^{N}\mathrm{e}^{j\beta_{q+i-1}}\mathbf{A}_q\right)\tilde{\boldsymbol{\phi}}_{\mathrm{s}}=\mathbf{0},\quad l=1,2,\ldots,L \tag{3.32a}$$

and

$$\mathbf{A}_{\mathrm{sc}}^{(1)} = [\lambda^2 \mathbf{M}_{\mathrm{cs}}^{(1)} + \lambda(\mathbf{C}_{\mathrm{cs}}^{(1)} - \mathbf{G}_{\mathrm{cs}}^{(1)}) + \mathbf{K}_{\mathrm{cs}}^{(1)} - \mathbf{H}_{\mathrm{cs}}^{(1)}]^{\mathrm{T}}, \tag{3.32b}$$

where $\beta_{N+q} = \beta_q$ for positive integer q. The first term of equation (3.32*a*) reduces to

$$\mathbf{A}_{\mathrm{sc}}^{(1)}(\mathbf{R}_i^{\mathrm{T}}\boldsymbol{\phi}_1^{(t)}, \mathbf{R}_i^{\mathrm{T}}\boldsymbol{\phi}_2^{(t)}, \ldots, \mathbf{R}_i^{\mathrm{T}}\boldsymbol{\phi}_P^{(t)})^{\mathrm{T}} \tag{3.33a}$$

and

$$\mathbf{R}_i^{\mathrm{T}}\boldsymbol{\phi}_p^{(t)} = \begin{pmatrix} \cos\beta_i & \sin\beta_i & 0 \\ -\sin\beta_i & \cos\beta_i & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_p \\ jx_p \\ 0 \end{pmatrix} = \begin{pmatrix} x_p \mathrm{e}^{j\beta_i} \\ jx_p \mathrm{e}^{j\beta_i} \\ 0 \end{pmatrix}, \quad i = 1, 2, \ldots, N, \ \ p = 1, 2, \ldots, P. \tag{3.33b}$$

With equation (3.33), multiplication of equation (3.32*a*) by $\mathrm{e}^{-j\beta_i}$ yields (because $\beta_i = 2(i-1)\pi/N$)

$$\mathbf{A}_{\mathrm{sc}}^{(1)}(\boldsymbol{\phi}_1^{(t)}, \boldsymbol{\phi}_2^{(t)}, \ldots, \boldsymbol{\phi}_P^{(t)})^{\mathrm{T}} + \left(\sum_{q=1}^{N} \mathrm{e}^{j\beta_{q-1}} \mathbf{A}_q\right) \tilde{\boldsymbol{\phi}}_{\mathrm{s}} = \mathbf{0}. \tag{3.34}$$

Equation (3.34) is independent of the index i. Thus, substitution of the proposed eigenvector for $k = 1$ into the equations that govern the ith substructure motions yields the same L equations in equation (3.34) for every $i = 1, 2, \ldots, N$.

Combining the P independent equations derived from the equations that govern the central component motions and the L equations in equation (3.34) obtained from the equations governing the substructure motions into a matrix expression yields a $(P+L) \times (P+L)$ reduced eigenvalue problem for $k = 1$. The matrices are complex-valued and depend on λ. The unknown degree-of-freedom vector consists of x_p $(p = 1, 2, \ldots, P)$ and elements of the L-dimensional vector $\tilde{\boldsymbol{\phi}}_{\mathrm{s}}$. This $(P+L) \times (P+L)$ reduced eigenvalue problem provides $P+L$ eigensolutions for $k = 1$. The full system eigenvectors are reconstructed from these $P+L$ eigenvectors of the reduced problem by using equations (3.2) and (3.20) and $\boldsymbol{\phi}_p^{(t)} = (x_p, jx_p, 0)^{\mathrm{T}}$.

When $k = N-1$, the proposed eigenvector is the complex conjugate of that for $k = 1$. Following the above process yields a $(P+L) \times (P+L)$ eigenvalue problem that is the complex conjugate of that just derived for $k = 1$. This complex conjugate reduced eigenvalue problem has eigensolutions that are the complex conjugates of the previous $P+L$ eigensolutions.

Thus, there are a total of $2(P+L)$ eigensolutions for $k = 1$ and $N-1$. Because these $2(P+L)$ eigensolutions have pure central component translations (no central component rotations), they are called *translational modes*. These eigensolutions are fully determined from one complex-valued $(P+L) \times (P+L)$ eigenvalue problem. There is no need to formulate or solve the full system eigenvalue problem in equation (3.3).

Rotational modes. The vibration modes when $k = 0$ in planetary gears [6,19–22] and CPVA systems [23–25] are rotational modes, where no central component translations exist. This motivates selection of a candidate mode where the central component translations vanish for $k = 0$. Thus, the motions of each central component are proposed as $\boldsymbol{\phi}_p^{(r)} = (0, 0, \mu_p)^{\mathrm{T}}$ for $p = 1, 2, \ldots, P$. Furthermore, for $k = 0$, $\hat{\mathbf{R}}$ in equation (3.22) is

$$\hat{\mathbf{R}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N \end{pmatrix}. \tag{3.35}$$

Substitution of the central component motions $\boldsymbol{\phi}_p^{(r)} = (0, 0, \mu_p)^{\mathrm{T}}$ into the first term in equation (3.21) and use of equation (3.35) in equation (3.22*a*), which is the second term of equation (3.21), reveal that the $2P$ x- and y-translational equations for the P central components vanish for $k = 0$. Thus, equation (3.21) yields only P independent equations generated from the equations that govern the central component rotations.

For $k = 0$, the proposed central component motion $\boldsymbol{\phi}_p^{(r)} = (0, 0, \mu_p)^{\mathrm{T}}$ is invariant under the rotation operation of $\mathbf{R}_i$, i.e. $\mathbf{R}_i\boldsymbol{\phi}_p^{(r)} = \boldsymbol{\phi}_p^{(r)}$ with $\mathbf{R}_i$ defined in equation (3.7). From equation (3.20), the proposed substructure motions for $k=0$ are $\boldsymbol{\phi}_s = (\underbrace{\tilde{\boldsymbol{\phi}}_s, \tilde{\boldsymbol{\phi}}_s, \ldots, \tilde{\boldsymbol{\phi}}_s}_{N})^T$, giving $\mathbf{A}_s^{(i)}\boldsymbol{\phi}_s = (\sum_{q=1}^{N}\mathbf{A}_q)\tilde{\boldsymbol{\phi}}_s$. Thus, following the procedure used for translational modes, substitution of the eigenvector for $k=0$ into the ith substructure equations in equation (3.3) yields the same L equations for all $i=1,2,\ldots,N$, which are

$$\mathbf{A}_{sc}^{(1)}(\boldsymbol{\phi}_1^{(r)}, \boldsymbol{\phi}_2^{(r)}, \ldots, \boldsymbol{\phi}_P^{(r)})^T + \left(\sum_{q=1}^{N}\mathbf{A}_q\right)\tilde{\boldsymbol{\phi}}_s = \mathbf{0}, \quad l=1,2,\ldots,L. \tag{3.36}$$

Therefore, we get a total of L independent equations from the substructure equations.

Combining the P equations from equation (3.21) and the L equations from equation (3.36) reveals that substitution of the proposed eigenvector (that is, equations (3.2) and (3.20) and $\boldsymbol{\phi}_p^{(r)} = (0, 0, \mu_p)^T$) for $k=0$ into the eigenvalue problem in equation (3.3) yields $P+L$ independent equations. Forming these $P+L$ equations into matrix form gives a $(P+L)\times(P+L)$ real-valued, reduced eigenvalue problem that provides $P+L$ eigensolutions for $k=0$. Because these $P+L$ eigensolutions have pure central component rotations (with no central component translations), they are called *rotational modes*. As with the prior mode types, they are determined from an eigenvalue problem that is much smaller and less computationally expensive than equation (3.3).

(iii) Completeness of the modal decomposition

The above derivations reveal that there are $(N-3)L$ substructure modes, $2(P+L)$ translational modes and $P+L$ rotational modes for systems with uncoupled central component translations and rotations, giving a total of $3P+NL$ modes. The $2(P+L)$ translational modes and $P+L$ rotational modes become $3(P+L)$ coupled modes for more general cyclically symmetric systems with coupling between central component translation and rotation. This total equals the total degrees of freedom of the system. Therefore, the eigenspace is completely determined by the substructure, translational and rotational modes derived above. No other mode type exists.

As shown in equation (3.20), for a given mode the substructures have the same modal deflection determined by the vector $\tilde{\boldsymbol{\phi}}_s$ in all of the three mode types. The phases of the substructure motions are determined by the phase index k and the spacing of the substructures β_i. For substructure modes, the phase indices are $k=2,3,\ldots,N-2$. The phase indices of translational modes are $k=1$ and $N-1$. For rotational modes, $k=0$.

4. Modal structure of conservative, non-gyroscopic systems

For conservative, non-gyroscopic systems, the eigenvalue problem is

$$\lambda^2\mathbf{M}\boldsymbol{\phi} + \mathbf{K}\boldsymbol{\phi} = \mathbf{0}, \tag{4.1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are symmetric mass and stiffness matrices, respectively. The eigenvalues (which for this section means λ^2 instead of λ) and eigenvectors of these systems are real for conservative, non-gyroscopic systems [35]. Because the modal property derivations above remain valid for $\mathbf{G}=\mathbf{C}=\mathbf{H}=\mathbf{0}$, the eigenvectors in equation (3.2) (whose substructure motions are given in (3.20)) must satisfy equation (4.1). The earlier derivation shows that all substructure modes except for $k=N/2$ (even N) and all translational modes are complex-valued. Inspection of equation (4.1) for complex-valued $\boldsymbol{\phi}$ and real-valued λ^2 shows that both the real and imaginary parts of $\boldsymbol{\phi}$ satisfy equation (4.1), which is consistent with the fact that there exist real-valued eigenvectors of equation (4.1) for symmetric $\mathbf{M}$ and $\mathbf{K}$. These real and imaginary parts of the eigenvector in equation (3.2) associate with the same eigenvalue. The eigenvalue λ^2 has two independent eigenvectors and so is degenerate with multiplicity two. Therefore, for conservative, non-gyroscopic systems, all substructure modes with $k\neq N/2$ and all translational modes are degenerate. The complex conjugate of $\boldsymbol{\phi}$ gives the same two degenerate real-valued eigenvectors.

As discussed above, the real and imaginary parts of the substructure mode eigenvector $\boldsymbol{\phi}$ in equation (3.24) form two degenerate vibration modes ($k \neq N/2$), which are

$$\hat{\boldsymbol{\phi}} = (\mathbf{0}_{1\times 3P}, \hat{\boldsymbol{\phi}}_1^{(s)}, \hat{\boldsymbol{\phi}}_2^{(s)}, \ldots, \hat{\boldsymbol{\phi}}_N^{(s)})^{\mathrm{T}}, \quad \hat{\boldsymbol{\phi}}_i^{(s)} = \cos k\beta_i \tilde{\boldsymbol{\phi}}_{\mathrm{s}} \tag{4.2a}$$

and

$$\check{\boldsymbol{\phi}} = (\mathbf{0}_{1\times 3P}, \check{\boldsymbol{\phi}}_1^{(s)}, \check{\boldsymbol{\phi}}_2^{(s)}, \ldots, \check{\boldsymbol{\phi}}_N^{(s)})^{\mathrm{T}}, \quad \check{\boldsymbol{\phi}}_i^{(s)} = \sin k\beta_i \tilde{\boldsymbol{\phi}}_{\mathrm{s}}. \tag{4.2b}$$

The complex conjugate of $\boldsymbol{\phi}$ in equation (3.24) forms the same pair of degenerate real-valued eigenvectors in equation (4.2). Because $\beta_1 = 0$, the substructure motions of the two real-valued substructure modes in equation (4.2) satisfy

$$\begin{aligned}\begin{pmatrix}\hat{\boldsymbol{\phi}}_i^{(s)}\\ \check{\boldsymbol{\phi}}_i^{(s)}\end{pmatrix} &= \begin{pmatrix}\cos k\beta_i \tilde{\boldsymbol{\phi}}_{\mathrm{s}}\\ \sin k\beta_i \tilde{\boldsymbol{\phi}}_{\mathrm{s}}\end{pmatrix} = \begin{pmatrix}\cos k(\beta_i+\beta_1)\tilde{\boldsymbol{\phi}}_{\mathrm{s}}\\ \sin k(\beta_i+\beta_1)\tilde{\boldsymbol{\phi}}_{\mathrm{s}}\end{pmatrix}\\
&= \begin{pmatrix}\cos k\beta_i \mathbf{I}_{L\times L} & -\sin k\beta_i \mathbf{I}_{L\times L}\\ \sin k\beta_i \mathbf{I}_{L\times L} & \cos k\beta_i \mathbf{I}_{L\times L}\end{pmatrix}\begin{pmatrix}\cos k\beta_1 \tilde{\boldsymbol{\phi}}_{\mathrm{s}}\\ \sin k\beta_1 \tilde{\boldsymbol{\phi}}_{\mathrm{s}}\end{pmatrix}\\
&= \begin{pmatrix}\cos k\beta_i \mathbf{I}_{L\times L} & -\sin k\beta_i \mathbf{I}_{L\times L}\\ \sin k\beta_i \mathbf{I}_{L\times L} & \cos k\beta_i \mathbf{I}_{L\times L}\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{\phi}}_1^{(s)}\\ \check{\boldsymbol{\phi}}_1^{(s)}\end{pmatrix},\\
& k = 2, 3, \ldots, N-2, \quad k \neq \frac{N}{2}.\end{aligned} \tag{4.3}$$

This relates the ith substructure motions to the first substructure motions. For $k = N/2$ (even N), the substructure mode in equation (3.24) becomes

$$\boldsymbol{\phi} = (\mathbf{0}_{1\times 3P}, \tilde{\boldsymbol{\phi}}_{\mathrm{s}}, -\tilde{\boldsymbol{\phi}}_{\mathrm{s}}, \tilde{\boldsymbol{\phi}}_{\mathrm{s}}, \ldots, -\tilde{\boldsymbol{\phi}}_{\mathrm{s}})^{\mathrm{T}}, \tag{4.4}$$

where $\tilde{\boldsymbol{\phi}}_{\mathrm{s}}$ is real-valued for conservative, non-gyroscopic systems. These real-valued substructure modes have distinct eigenvalues. Combining the degenerate and distinct substructure modes with $k = 2, 3, \ldots, N-2$, the total number of substructure modes is $(N-3)L$, the same as derived for general systems.

The modes associated with $k = 1$ and $N-1$ are translational modes. They are degenerate with multiplicity two with real-valued eigenvectors. The substructure motions of these degenerate translational modes satisfy equation (4.3) for $k = 1$ and $N-1$. The complex-valued central component motion in equation (3.2) for translational modes can be expressed as

$$\boldsymbol{\phi}_p^{(t)} = (x_p, jx_p, 0)^{\mathrm{T}} = (a_p + jb_p, -b_p + ja_p, 0)^{\mathrm{T}}, \quad p = 1, 2, \ldots, P. \tag{4.5}$$

The real-valued eigenvectors come from the real and imaginary parts of the complex translational mode. Thus, the central component motions of the two degenerate real-valued translational modes are

$$\hat{\boldsymbol{\phi}}_p^{(t)} = (a_p, -b_p, 0)^{\mathrm{T}} \quad \text{and} \quad \check{\boldsymbol{\phi}}_p^{(t)} = (b_p, a_p, 0)^{\mathrm{T}}. \tag{4.6}$$

There are $P+L$ degenerate translational modes with multiplicity two. Therefore, in total, there are $2(P+L)$ independent translational mode eigenvectors, the same as for general systems.

For rotational modes ($k = 0$), the eigenvectors from equations (3.2) and (3.20) are real. The rotational modes are distinct and their properties in the conservative, non-gyroscopic case remain identical to those derived for general systems. $P+L$ distinct rotational modes exist in the system.

5. Numerical solutions for the example system

Table 1 shows the eigenvalues of the example system in figure 2 with three, four and five substructures, and the system parameters given by table 2 for $\Omega = 5000$ r.p.m. (523.60 rad s^{-1}). Each of the three mode types shown in table 1 includes a degenerate eigenvalue with multiplicity two. This degeneracy is specific for the chosen example system; for general cyclically symmetric

Table 1. Natural frequencies, $\omega = \mathrm{Im}(\lambda)$ (rad s^{-1}), for the example cyclically symmetric system in figure 2 with N equally spaced, identical substructures at rotating speed $\Omega = 523.60$ rad s^{-1}. The system parameters are given in table 2. Multiplicities are shown in parentheses.

mode type	$N=3$	$N=4$	$N=5$
substructure	n.a.	0	6.6705 (2)
		1398.5	1389.8
			1403.2
translational	8.6631 (2)	11.581 (2)	10.356 (2)
	1391.1	1385.9	1387.7
	1408.4	1409.1	1408.4
	6554.5	6556.9	6559.2
	7601.7	7604.1	7606.4
rotational	0 (2)	0 (2)	0 (2)
	1428.1	1443.0	1456.2

Table 2. Parameters of cyclically symmetric example system in figure 2.

parameter	value
central component mass, m_c (kg)	20
central component inertia, J_c (kg m^2)	0.2
central component bearing stiffness, k_c (N m^{-1})	10^9
stiffness between central component and substructure, k_{cs} (N m^{-1})	10^6
stiffness between neighbouring substructures, k_s (N m^{-1})	10×10^3
substructure mass, m (kg)	0.9
distance between axis of symmetry and substructure equilibrium position, l (m)	0.04

systems, this degeneracy is not expected. In planetary gears [6,19–22,34] and CPVA systems [23–25], for example, this eigenvalue degeneracy does not occur.

The number of substructure modes derived analytically is $(N-3)L$. Table 1 confirms this result with $L=2$. Figure 4*a*,*b* shows the complex-valued substructure mode with $\lambda = 1403.2j$ rad s^{-1} at $\Omega = 5000$ r.p.m. for a system with five substructures and the system parameters given by table 2. No central component motions exist in the substructure mode. The substructures move with equal amplitudes. Their phases increase sequentially by $k\beta_i$ for $k = 2, 3, \ldots, N-2$ ($k=2$ for the substructure mode shown in figure 4*a*,*b*). These properties match the substructure mode given in equation (3.24).

There are $2(P+L)$ translational modes for general cyclically symmetric systems. This is confirmed by the number of translational modes of the example system shown in table 1 for $P=1$ and $L=2$. Figure 4*c*,*d* shows the translational mode with $\lambda = 7606.4j$ rad s^{-1} at $\Omega = 5000$ r.p.m. for a system with five substructures and the system parameters given by table 2. The central component has pure translation (no rotation). The two central component translational degrees of freedom move with equal amplitudes and they are 90° out of phase, consistent with the form $\boldsymbol{\phi}_p^{(t)} = (x_p, jx_p, 0)^{\mathrm{T}}$. The substructure motions have equal amplitudes. Their phases increase

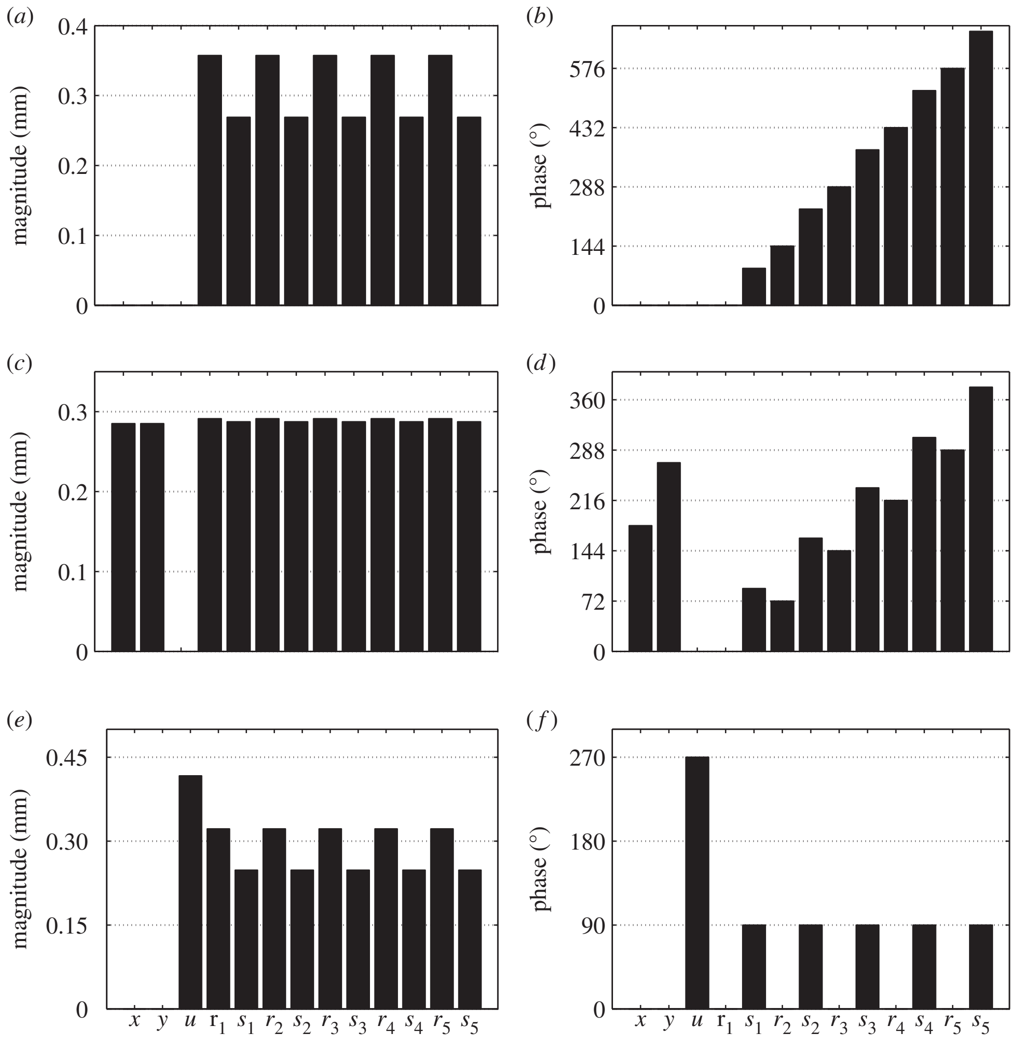

Figure 4. Structured vibration modes of the example cyclically symmetric system in figure 2 with five equally spaced, identical substructures and the system parameters given in table 2 at rotation speed $\Omega = 5000$ r.p.m. (523.60 rad s^{-1}). The horizontal axis labels denote the system degrees of freedom. (*a*) Substructure mode magnitude, $\lambda = 1403.2j$ rad s^{-1}, $k = 2$. (*b*) Substructure mode phase, $\lambda = 1403.2j$ rad s^{-1}, $k = 2$. (*c*) Translational mode magnitude, $\lambda = 7606.4j$ rad s^{-1}, $k = 1$. (*d*) Translational mode phase, $\lambda = 7606.4j$ rad s^{-1}, $k = 1$. (*e*) Rotational mode magnitude, $\lambda = 1456.2j$ rad s^{-1}. (*f*) Rotational mode phase, $\lambda = 1456.2j$ rad s^{-1}.

sequentially by β_i because the translational mode shown in figure 4*c*,*d* associates with $k = 1$. The central component x-translation and the first substructure radial degree of freedom r_1 are 180° out of phase, as shown in figure 4*d*. This phase relation is also found in CPVA systems [23–25]. For the translational modes associated with $k = N - 1$, the phases of the substructure motions decrease sequentially by β_i.

There are three rotational modes shown in table 1, equalling the number of rotational modes $(P + L)$ derived for general cyclically symmetric systems. Figure 4*e*,*f* shows the rotational mode with $\lambda = 1456.2j$ rad s^{-1} at $\Omega = 5000$ r.p.m. for a system with five substructures and the system parameters given by table 1. The central component has pure rotation (no translation). The substructures move identically, that is, in phase with each other and with equal amplitudes. Similar to the phase relation between the x-translation and r_1 in translational modes, the central

Table 3. Natural frequencies, $\omega = \mathrm{Im}(\lambda)$ (rad s^{-1}), for the example cyclically symmetric system in figure 2 with N equally spaced, identical substructures at rotating speed $\Omega = 0$ rad s^{-1}. The system parameters are given in table 2. Multiplicities are shown in parentheses.

mode type	$N=3$	$N=4$	$N=5$
substructure	n.a.	0	0 (2)
		1064.6	1062.0 (2)
translational	0 (2)	0 (2)	0 (2)
	1066.4 (2)	1063.6 (2)	1064.3 (2)
	7076.8 (2)	7078.7 (2)	7080.6 (2)
rotational	0 (2)	0 (2)	0 (2)
	1059.3	1064.6	1067.8

component rotation is 180° out of phase with the substructure tangential degrees of freedom in this rotational mode. The radial and tangential degrees of freedom for each substructure are 90° out of phase. This phase relation is similar to the central component x- and y-translations of translational modes.

For the conservative, non-gyroscopic case where $\Omega = 0$, the analytical results predict the substructure mode eigenvalue for $N = 5$ and the translational mode eigenvalues for $N = 3, 4, 5$ are degenerate with multiplicity two. The numerical results in table 3 confirm this. For $N = 4$, exactly one distinct substructure mode associated with $k = N/2 = 2$ is obtained, as predicted analytically. The numerical planetary gear example given in [19,22] for zero speed also gives degenerate translational modes with multiplicity two. For the planet modes (i.e. the substructure modes) of planetary gears, the eigenvalue multiplicity is $N - 3$ [19,22] because the planet gears (substructures in planetary gears) do not connect with each other directly, as discussed above as a special case.

Considering the degenerate substructure modes associated with $\omega = 1062.0$ rad s^{-1} for the example system with five substructures and the system parameters given by table 2 at zero rotation speed, the motions of the first substructure in the two degenerate substructure modes in equation (4.2) are $\hat{\boldsymbol{\phi}}_1^{(s)} = (0.6271, 0.0009)^{\mathrm{T}}$ and $\check{\boldsymbol{\phi}}_1^{(s)} = (0.0822, -0.0069)^{\mathrm{T}}$. The motions of the second substructure in these two degenerate substructure modes are $\hat{\boldsymbol{\phi}}_2^{(s)} = (-0.4590, -0.0048)^{\mathrm{T}}$ and $\check{\boldsymbol{\phi}}_2^{(s)} = (-0.4351, 0.0051)^{\mathrm{T}}$. These substructure motions satisfy equation (4.3) with $k = 2$ and $\beta_2 = 2\pi/5$, and so do the other substructure motions in these two substructure modes.

The central component motions in two degenerate translational modes associated with $\omega = 1064.3$ rad s^{-1} of the example system with five substructures at zero rotation speed are $\hat{\boldsymbol{\phi}}^{(c)} = (0.0017, 0, 0)^{\mathrm{T}}$ and $\check{\boldsymbol{\phi}}^{(c)} = (0, 0.0017, 0)^{\mathrm{T}}$; these satisfy equation (4.6). The motions of the first substructure in these two degenerate translational modes are $\hat{\boldsymbol{\phi}}_1^{(s)} = (0.6324, 0)^{\mathrm{T}}$ and $\check{\boldsymbol{\phi}}_1^{(s)} = (0, 0.0095)^{\mathrm{T}}$, and the motions of the second substructure are $\hat{\boldsymbol{\phi}}_2^{(s)} = (0.1954, -0.0091)^{\mathrm{T}}$ and $\check{\boldsymbol{\phi}}_2^{(s)} = (0.6014, 0.0029)^{\mathrm{T}}$. These substructure motions satisfy equation (4.3) with $k = 1$.

6. Conclusion

A structured modal decomposition of general cyclically symmetric systems is derived in this work. The analysis of cyclically symmetric systems with non-vibrating central components reveals that the eigenvectors are characterized by phase indices that determine the phase relations between the cyclically symmetric substructures. We show that this property remains valid for cyclically symmetric systems with central component vibrations. Furthermore, the vibration mode structure consists of substructure and coupled modes. For systems with uncoupled central component translations and rotations, which is the usual case, the eigenvectors are categorized into exactly three different mode types (substructure, translational and rotational) with specific central component motions, phase indices, substructure motion features and eigenvalue degeneracy for each type. Additional properties are determined for the conservative, non-gyroscopic case.

The analysis reveals properties of the system matrices that dramatically simplify the modelling complexity. The necessary matrices, from which the full system matrices are readily constructed, are available from three simplified systems that include: (i) the first substructure and any substructures to which it connects, (ii) the first substructure and any central components to which it connects, and (iii) only the central components. This advantage applies for both finite-element modelling or when deriving the equations of motion directly.

All modes in each of the mode types can be calculated from much smaller eigenvalue problems than the full system eigenvalue problem, giving substantial computational advantage for systems with many substructures and/or many degrees of freedom per substructure (e.g. bladed disc assemblies). The necessary matrices for these reduced eigenvalue problems are exactly those derived from three simplified systems mentioned above. Alternatively, they can be directly extracted from the full system matrices if those are available.

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Appendix A. Derivation of equation (3.32*a*)

From equations (3.2) and (3.3), the substructure equations are $\mathbf{A}_{sc}\boldsymbol{\phi}_c + \mathbf{A}_s\boldsymbol{\phi}_s = \mathbf{0}$. Considering only the first term (and only the mass matrix), we partition $\mathbf{M}_{sc}$ as

$$\mathbf{M}_{sc} = \begin{pmatrix} \mathbf{M}_{sc}^{(1)} \\ \mathbf{M}_{sc}^{(2)} \\ \vdots \\ \mathbf{M}_{sc}^{(N)} \end{pmatrix}. \tag{A 1}$$

Symmetry of the mass matrix ensures $\mathbf{M}_{sc} = \mathbf{M}_{cs}^{\mathrm{T}}$. This result, equation (A 1), and equation (3.6) give $\mathbf{M}_{sc}^{(i)} = [\mathbf{M}_{cs}^{(i)}]^{\mathrm{T}}$. Finally, with equation (3.7*a*), the relevant part of the mass matrix for the *i*th substructure's governing equations (i.e. equation (3.32*a*)) is

$$\mathbf{M}_{sc}^{(i)} = [\mathbf{M}_{cs}^{(i)}]^{\mathrm{T}} = [\mathbf{M}_{cs}^{(1)}]^{\mathrm{T}}\mathrm{diag}(\mathbf{R}_i^{\mathrm{T}}, \mathbf{R}_i^{\mathrm{T}}, \ldots, \mathbf{R}_i^{\mathrm{T}}). \tag{A 2}$$

Because of their skew-symmetry, a minus sign is introduced in equation (3.32*b*) for the gyroscopic ($\mathbf{G}_{sc}^{(i)} = -[\mathbf{G}_{cs}^{(i)}]^{\mathrm{T}}$) and circulatory matrices.

The second term of equation (3.32*a*) comes from $\mathbf{A}_s\boldsymbol{\phi}_s$. It is reduced from $\mathbf{A}_s^{(i)}\boldsymbol{\phi}_s$, where $\mathbf{A}_s^{(i)}$ is the *i*th $N \times NL$ submatrix of the $NL \times NL$ block circulant matrix $\mathbf{A}_s$ (the first two submatrices are in equations (2.3) and (2.4)) and $\tilde{\boldsymbol{\phi}}_s$ is from the partitioned substructure motion in equations (3.2) and (3.20).

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