Dispersive equations with random initial data

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June 18, 2018

Nonlinear Schrödinger Equation

Semilinear initial value problem:

$$\begin{aligned} i\partial_t u(t,x) - \partial_x^2 u(t,x) &= \pm(|u|^2 u)(t,x), \\ u(0,x) &= u_0(x). \end{aligned}$$

Here $x \in \mathbb{T} = [0, 2\pi)$.

Can also consider the equation on $\mathbb{R}$, $\mathbb{T}^n$, $\mathbb{R}^n$.

NLS appears in the study of several wave propagation phenomena, for example (in the asymptotic limit) in the desctiption of small amplitude water waves.

Zakharov 1968: Hamiltonian formulation of irrotational water wave problem.

Totz-Wu 2012: rigorous derivation of modulation approximation by NLS.

Smooth initial data

Reformulate as integral equation

$$u(t,x) = (e^{it\partial_x^2} u_0)(x) + \int_0^t e^{i(t-s)\partial_x^2} (|u|^2 u)(s,x)\,\mathrm{d}s. \qquad (1)$$

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If initial data has $s > 1/2$ ($s > n/2$ in dimension n) derivatives in L^2:

$$\|u_0\|_{H^s}^2 = \sum_{n\in\mathbb{Z}} |n|^{2s} |\widehat{u_0}(n)|^2.$$

Smooth initial data

The right side of (1)

$$\int_0^t e^{i(t-s)\partial_x^2}(|u|^2 u)(s,x)\,\mathrm{d}s$$

is less than

$$\|e^{it\partial_x^2}u_0\|_{L^2} + Ct\sup_{s\le t}\|u(s,\cdot)\|_{H^s}^3.$$

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Can apply contraction mapping principle.

Less regular initial data

Long series of work on less regular initial data: Ginibre-Velo, Cazenave-Weissler, Kenig-Ponce-Vega, Bourgain,... Using delicate techniques from harmonic analysis. (Strichartz estimates, multilinear estimates, restriction spaces.)

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Reason to consider initial data with few derivatives: corresponds to the regularity of quantities such as the *Hamiltonian* ("energy"):

$$H(u) = \frac{1}{2}\int |\partial_x u|^2 \, \mathrm{d}x \mp \frac{1}{4}\int |u|^4 \, \mathrm{d}x.$$

Hamiltonian formulation

The equation can be formally written as

$$i\partial_t u = DH(u).$$

The quantity is on the right is the functional gradient:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta} H(u+\delta v)\Big|_{\delta=0} &= -\int \overline{\partial_{xx} u} v \,\mathrm{d}x \mp \int \overline{|u|^2 u} v \,\mathrm{d}x \\ &= \int \overline{DH(u)} v \,\mathrm{d}x.\end{aligned}$$

Hamiltonian formulation

Gradient is

$$DH(u) = -\partial_x^2 u \mp |u|^2 u.$$

We can think of NLS as the differential equation:

$$\dot{\Phi}_t(u) = -iDH(\Phi_t(u)), \tag{2}$$

where the right side is a *Hamiltonian vector field*.

Conservation of energy

The nonlinear flow $\Phi(u_0)(t) = u(t)$ preserves the Hamiltonian ("energy"):

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2 \,\mathrm{d}x \mp \frac{1}{4}\int_{\mathbb{T}} |u|^4 \,\mathrm{d}x.$$

Conservation of energy

Formally,

$$\frac{\mathrm{d}}{\mathrm{d}t}H(u) = i\int \overline{DH(u)}DH(u)\,\mathrm{d}x.$$

Right side is imaginary, $H(u)$ is real-valued, so $\frac{\mathrm{d}}{\mathrm{d}t}H(u) = 0$.

$$H(u(0)) = H(u(t)),$$

"Energy is conserved".

Another consequence: Conservation of “mass”

Take $F(u) = \|u\|_{L^2}^2$, and compute

$$\frac{\mathrm{d}}{\mathrm{d}t} F(u(t)) = i \int \overline{DH(u)} u \,\mathrm{d}x.$$

The right side is imaginary, but $F(u)$ is real-valued, so $\frac{\mathrm{d}}{\mathrm{d}t} F(u) = 0$: the L^2 norm is conserved.

Long time behavior

If the initial data is sufficiently regular for $H(u)$ to make sense, conservation of energy extends the solution to all times.

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With plus $(+)$ sign ("defocusing equation"), if the initial data is in H^1, then

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If we can construct local solutions with inital data in H^1 with time of existence $= T(\|u_0\|_{H^1})$, then we get global solutions. (Ginibre-Velo, 1980s.)

Long time behavior

With $(-)$ sign ("focusing equation"), there is a competition between the two terms in H:

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2 \,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{T}} |u|^4 \,\mathrm{d}x.$$

Long time behavior: Sobolev inequality

On $\mathbb{R}^n$ (or $\mathbb{T}^n$):

$$\|u\|_{L^{p+1}} \lesssim \|u\|_{\dot{H}^s} \lesssim \|\nabla u\|_{L^2}^{s} \|u\|_{L^2}^{1-s}, \tag{3}$$

with $s = \frac{n(p-1)}{2(p+1)}$ so the gradient squared dominates provided $s(p+1) < 2 \Rightarrow 3 = p < 1 + \frac{4}{n}$.

Long time behavior

- For high ("critical", "supercritical") nonlinearities, the question of long term behavior is very delicate. It has been known for a long time that solutions can die in finite time, but descriptions of the blowup are difficult to obtain.
- Long line of research on blowup solutions: Strauss, Weinstein, Kohn, Martel, Merle, Raphael, Lions...
- One starting point is to examine equation (3) more carefully. (Weinstein)

Invariance of measure by Hamiltonian flows

Recall the Hamiltonian formulation of NLS:

$$\dot{\Phi}_t = -iDH(\Phi_t).$$

In finite dimensions, a Hamiltonian equation has the form:

$$\dot{\Phi}_t(p,q) = X_H(\Phi_t(p,q)), \quad (p,q) \in \mathbb{R}^{n \times n}.$$

$$\begin{aligned} X_H(p,q) &= \begin{pmatrix} & I \\ -I & \end{pmatrix} \nabla_{p,q} H \\ &= (\partial_{q_1} H, \dots, \partial_{q_n} H, -\partial_{p_1} H, \dots, -\partial_{p_n} H). \end{aligned}$$

Conservation of H

Also in finite dimensions, we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(p,q) &= \nabla H \cdot J \nabla H \\ &= \sum_{j=1}^{n}\left(-\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial q_j} + \frac{\partial H}{\partial p_j}\frac{\partial H}{\partial q_j}\right) \\ &= 0.\end{aligned}$$

“Energy is conserved.”

Liouville theorem

If $\dot{\Phi}_t = X_H(\Phi_t)$, then the Lebesgue measure $\mathrm{d}p\mathrm{d}q$ is invariant:

$$\int f(\Phi_t(p,q))\,\mathrm{d}p\mathrm{d}q = \int f(p,q)\,\mathrm{d}p\mathrm{d}q.$$

This follows because X_H is divergence-free.

Liouville theorem

If $F(p,q)$ is invariant: $F(p(t),q(t)) = F(p_0,q_0)$, then $F(p,q)\,\mathrm{d}p\mathrm{d}q$ is also invariant.

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Example: $F(p, q) = e^{-\beta H(p,q)}$, is an invariant, "Gibbs", measure.

Based on an analogy with finite-dimensional Hamiltonian dynamics, Lebowitz-Rose-Speer (1987) suggested that the "Gibbs-type" measure

$$\text{“}e^{-H(u)}\,\mathrm{d}u\text{”}. \tag{4}$$

is *invariant* under flow of the NLS equation: $i\partial_t u = \partial_{xx} u \pm |u|^2 u$.

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If $F(u(x), x \in \mathbb{T})$ is a function of the path, then

$$\mathbb{E}F(u_0) = \mathbb{E}F(u(t)).$$

How to make sense of (4)?

Motivation

If we can make sence of the measure, we can use the invariant measure to control the solutions.

Lebowitz-Rose-Speer: choose u_0 at random according to the Gibbs measure, is $u(t) = \Phi_t(u_0)$ distributed according to Gibbs?

Dynamics is deterministic: different from stochastic differential equations.

Invariance implies global existence

Theorem (Bourgain, 1993)

If u_0 is distributed according to

$$\mu(\cdot) = \frac{1}{Z}\mathbb{E}_{\mu_1}[e^{\frac{1}{4}\int_{\mathbb{T}} |u|^4 \,\mathrm{d}x}(\cdot)\underbrace{\mathbf{1}[\|u\|_2 \leq K]}_{\text{cutoff}}],$$

there is almost surely a global solution of

$$iu_t + u_{xx} + u|u|^2 = 0,$$

with initial data u_0 in $C(\mathbb{R}_+, H^s)$, $s < 1/2$. Moreover, the distribution of $u(t,\cdot)$ is invariant under μ.

Invariance

1. Bourgain introduced $X^{s,b}(\mathbb{T})$ spaces to prove local-wellposedness in a space of measure 1 for μ_1 (H^s, $s < 1/2$) .
2. Uses invariance to obtain global solutions.
3. Started a large field of study: Bourgain, Burq, Chatterjee, Thomann, Tzvetkov, Staffilani, Oh, etc.

Consequence of invariance

To prove his theorem on existence of invariant solutions, Bourgain showed how to use the existence of the invariant measure as a replacement for a global solution. His solutions satisfy:

$$\frac{\|u(t)\|_{H^s}}{\sqrt{\log(1+t)}} < \infty$$

with $0 < s < 1/2$. This was not known for deterministic solutions.

Estimating the growth of smooth, global solutions of dispersive equations is a major challenge.

Problem Lebowitz-Rose-Speer face: how to define

$$e^{-H(u)}\,\mathrm{d}u?$$

Construction: quadratic part

Very old idea: to make sense of $e^{-H(u)}\mathrm{d}u$, separate the Hamiltonian into two parts:

$$\underbrace{e^{-\frac{1}{2}\int |\partial_x u|^2\,\mathrm{d}x}}_{\text{quadratic part}} e^{\pm\frac{1}{4}\int |u|^4\,\mathrm{d}x}.$$

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Fourier Transform:

$$\begin{aligned} e^{-\frac{1}{2}\int |\partial_x u|^2\,\mathrm{d}x} &= e^{-\frac{1}{2}\sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^2|\widehat{u}(n)|^2} \\ &= \prod_{n\neq 0} e^{-|n|^2|\widehat{u}(n)|^2}. \end{aligned}$$

The quadratic part is a measure μ: independent Gaussians along the different frequencies.

Construction: quadratic part

μ is the distribution of the random series:

$$u(x) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{g_n}{|n|} e^{inx},$$

where

$$g_n = \Re g_n + i \Im g_n,$$

independent Gaussians with variance $1/2$.

Construction: quadratic part

The series for u converges in H^σ, $\sigma < 1/2$. If $u_N = \sum_{|n|\leq N} \frac{g_n}{|n|} e^{inx}$,

$$\sup_{N,N'} \mathbb{E}\|u_N - u_{N'}\|_{H^\sigma}^2 = \sum_{|n|=N'}^{N} \frac{1}{|n|^{2(1-\sigma)}} \to 0.$$

So the limiting measure μ_s makes sense on H^σ.

Construction: quadratic part

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On the other hand:

$$\mathbb{E}\|u_N\|^2_{H^s} = \sum_{|n| \leq N} 1 \to \infty.$$

u has less than $1/2$ derivatives in L^2 (and any other L^p).
Hamiltonian does not actually make sense for u!

Construction: quadratic part

Random function

$$u(x) = \sum_{n \neq 0} \frac{g_n}{|n|} e^{inx}$$

is a periodic version of a Brownian motion. We call u the (zero mean) *Brownian loop*.

Analogy: random series for Brownian bridge on $[0, 2\pi]$:

$$B_t = \sum_{n \geq 1} g_n \frac{\cos(nt) - 1}{n} + g_n' \frac{\sin(nt)}{n}.$$

Nonlinear Gibbs measure

Back to Lebowitz-Rose-Speer. How to interpret "$e^{-H(u)}\,\mathrm{d}u$"?

We understand the "quadratic part" of Hamiltonian:

$$\frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2\,\mathrm{d}x = \|u\|_{H^1}^2.$$

So we guess:

$$\text{"}\int_A e^{-H(u)}\,\mathrm{d}u\text{"} = \frac{1}{Z}\mathbb{E}_\mu[e^{\frac{1}{4}\int_{\mathbb{T}}|u|^4}\mathbf{1}_A].$$

Nonlinear Gibbs measure

To build the measure μ_p, $p \geq 2$, we choose a function at random "with probability $\sim e^{-\frac{1}{2}\int |\partial_x u|^2 \, \mathrm{d}x}$" (the Brownian loop), weighted by the factor

$$\frac{e^{\mp \frac{1}{p+1} \int |u(x)|^{p+1} \, \mathrm{d}x}}{\text{Normalization}}.$$

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More formally, for a set of functions $A \subset L^2(\mathbb{T})$,

$$\mu_p(A) = \frac{1}{Z_{p,\pm}} \mathbf{E}[e^{\frac{\pm 1}{p+1}\int_{\mathbb{T}} |u|^{p+1}\,\mathrm{d}x} \mathbf{1}_A].$$

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Does this make sense? Is $Z_{p,\pm} < \infty$?

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With minus $(-)$ sign:

$$\mathbf{E}[e^{\frac{-1}{p+1}\int_{\mathbb{T}} |u|^{p+1}\,\mathrm{d}x}] \leq 1.$$

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With minus $(-)$ sign:

$$\mathbf{E}[e^{\frac{-1}{p+1}\int_{\mathbb{T}} |u|^{p+1}\, \mathrm{d}x}] \leq 1.$$

With plus $(+)$ sign: $\int_{\mathbb{T}} |u|^{p+1}\, \mathrm{d}x \geq \frac{1}{(2\pi)^{\frac{p-1}{2}}} (\int_{\mathbb{T}} |u|^2 \, \mathrm{d}x)^{\frac{p+1}{2}}$, so

$$\mathbf{E}[e^{\frac{-1}{p+1}\int_{\mathbb{T}} |u|^{p+1}\, \mathrm{d}x}] = \infty.$$

Need to correct the definition.

Nonlinear Gibbs measure

Theorem (LRS, 1988)

1. *For* $p < 5$, $e^{\frac{1}{p+1}\int_{\mathbb{T}}|u|^{p+1}\,\mathrm{d}s}\mathbf{1}_{\{\|u\|_{L^2}\leq K\}}$ *is integrable for any* K*.*
2. *For* $p = 5$, *integrable only if* $K < \|Q\|_{L^2}$ *where* Q *("ground state soliton") is an optimizer for Gagliardo-Nirenberg inequality*
$$\|u\|_{L^6}^6\| \leq C_{\mathrm{GN}}\|u_x\|_{L^2}^2\|u\|_{L^2}^4.$$

L-R-S also state "if" direction for no. 2, but the proof is not complete in the periodic case (noticed by Carlen-Lebowitz-Fröhlich, 2013).

Another question of LRS, what if $K = \|Q\|_{L^2}$?.

Sharp integrability for Gibbs measure

We can prove the exact threshold:

Theorem (S. 2017)

$$\mathbf{E}\big[e^{\frac{1}{p+1}\int_{\mathbb{T}}|u|^{p+1}\,\mathrm{d}x}\mathbf{1}_{\|u\|_{L^2}\le K}\big] < \infty$$

if $K < \|Q\|_{L^2}$.

So the nonlinear Gibbs measure exists whenever $p < 5$.

Dimension 2

Proof also applies to Gibbs measures on *radial functions* on the disc in higher dimensions.
For example, $d = 2$ (constructed by N. Tzvetkov):

$$v(r) = \sum_{n=1}^{\infty} \frac{J_0(z_n r)}{\sqrt{\pi}\|J_0(z_n \cdot)\|_{L^2[0,1])}} \frac{g_n}{z_n}$$

$r^2 = x^2 + y^2$,
$J_0(r)$: Bessel function,
z_n, $n \geq 0$: eigenvalues of Dirichlet Laplacian.

Dimension 2

Consider

$$\tilde{Z}_{p,K} = \mathbf{E}[e^{\frac{1}{4}\int_{D_1} |v(r)|^4 \, \mathrm{d}x} \mathbf{1}_{\|v\|_{L^2(D_1)} \le K}].$$

Then:

1. $Z_{p,K} < \infty$ for any $K > 0$ when $p > 0$.
2. $Z_{p,K} < \infty$ if $K < \|\tilde{Q}\|_{L^2(D_1)}$, where $\tilde{Q}$ is the ground state soliton in dimension 2.

What is Q? Why does it appear?

Formally, $Z_p = \mathbf{E}[e^{\frac{1}{p+1}\int |u(x)|^{p+1}\,\mathrm{d}x}]$ equals

$$Z_p = \int \exp\left(-\frac{1}{2}\int |\partial_x u(x)|^2\,\mathrm{d}x + \frac{1}{p+1}\int |u(x)|^{p+1}\,\mathrm{d}x\right)\,\mathrm{d}u(x).$$

What is Q? Why does it appear?

Gagliardo-Nirenberg inequality:

$$\frac{1}{p+1}\int |u(x)|^{p+1}\,\mathrm{d}x \le \frac{C_{\mathrm{GNS}}}{2}\int |\partial_x u|^2\,\mathrm{d}x \cdot \left(\int |u(x)|^2\,\mathrm{d}x\right)^2.$$

Weinstein: on $\mathbb{R}^n$, this inequality is sharp if only if

$$u(x) = aQ(b(x-x_0))$$

for some $x_0 \in \mathbb{R}^n$, $a \neq 0$, $b > 0$.

The ground state soliton

For $p < 1 + \frac{4}{n}$, can solve the problem

$$\inf_{u \in H^1, \|u\|_{L^2} = m} H(u) < 0.$$

For $p = 1 + \frac{4}{n}$, this problem no longer makes sense. Instead look at

$$\inf_{u \in H^1} \frac{\|\nabla u\|_{L^2}^{\frac{n(p-1)}{2}} \|u\|_{L^2}^{\frac{(2-n)p+3)}{2}}}{\|u\|_{L^{p+1}}^{2(p+1)}}.$$

Weinstein showed in the 1980s that there is a positive, radial, decreasing Q such that this is attained.

The ground state soliton

Euler-Lagrange equation for the variational problem:

$$\Delta u(x) + u^p(x) = u(x).$$

Deep analysis of Kwong: solution is unique up to translations.

Back to Gagliardo-Nirenberg inequality:

$$\frac{1}{p+1}\int |u(x)|^{p+1}\,\mathrm{d}x \le \frac{C_{\mathrm{GNS}}}{2}\int |\partial_x u|^2\,\mathrm{d}x \cdot \left(\int |u(x)|^2\,\mathrm{d}x\right)^2.$$

Unique optimizer up to translation and scaling is Q, and

$$C_{\mathrm{GNS}} = \frac{p+1}{2}\|Q\|_{L^2}^{1-p}.$$

So if $K < \|Q\|_{L^2}$, expect quadratic part to dominate

$$\int \exp\left(-\frac{1}{2}\int |\partial_x u(x)|^2 \,\mathrm{d}x + \frac{1}{p+1}\int |u(x)|^{p+1}\,\mathrm{d}x\right)\,\mathrm{d}u(x).$$

To make this precise, need to reduce to some finite dimensional approximation and estimate the rate of approximation carefully.

Integrability at the ground state level

What if $K = \|Q\|_{L^2}$?

Theorem (Oh, S., Tolomeo)

$$Z_{+,p=5,K=\|Q\|_{L^2}} < \infty.$$

This corresponds to a "discontinuous phase transition".

Some idea of the proof

1. Stability result in L^2: if u is far from any rescaled or translated soliton $\|u - Q_{\delta,x_0}\|_{L^2} > \epsilon$, then Gagliardo-Nirenberg fails by some amount:

$$\|u\|_{L^6(\mathbb{T})} < (C_{\mathrm{GNS}} - \delta(\epsilon))\|\partial_x u\|^2_{L^2(\mathbb{T})}\|u\|^4_{L^2(\mathbb{T})}. \tag{5}$$

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Enough estimate

$$\mathbf{E}\left[\exp\left(\frac{1}{6}\int_{\mathbb{T}}|u|^6\,\mathrm{d}x\right)\mathbf{1}_{\|u\|^2_{L^2}\leq\|Q\|^2_{L^2}}\mathbf{1}_{N_\epsilon}\right].$$

N_ϵ is an ϵ-neighborhood of the manifold $\{Q_{\delta,x_0}, \delta > 0, x_0 \in \mathbb{T}\}$.

Some idea of the proof

2. If we were only computing the expectation over a neighborhood of a single soliton (say $x_0 = 0$, $\delta \ll 1$):

$$\int_{S_\epsilon} \exp\left(\frac{1}{6}\int_{\mathbb{T}} |Q+v|^6 \,\mathrm{d}x - \frac{1}{2}\int |\partial_x Q|^2 \,\mathrm{d}x + \langle Q'', v\rangle\right) \mathrm{d}v,$$

$$S_\epsilon = \{\|v\|_{L^2}^2 \leq \epsilon, \|u\|_{L^2}^2 \leq \|Q\|_{L^2}^2\}.$$

Q is an optimizer of the quantity in the exponential, so we expect this can be estimated (formally) by

$$\int_{\substack{\|v\|_{L^2}^2 \leq \epsilon, \\ \|u\|_{L^2}^2 \leq \|Q\|_{L^2}^2}} \exp\left(-\frac{1}{2}\langle Av, v\rangle_{H^1}\right) \mathrm{d}v,$$

where

$$I + A = I + P_{H^1}(1-\partial_x^2)^{-1}(\partial_x^2 + 5Q^4 - 1)$$

corresponds to the second variation of $H(u)$ at Q.

Some idea of the proof

We expect (after suitable approximations) that

$$\int_{\substack{\|v\|_{L^2}^2 \le \epsilon, \\ \|u\|_{L^2}^2 \le \|Q\|_{L^2}^2}} \exp(-\frac{1}{2}\langle Av, v\rangle)\,\mathrm{d}v$$

$$\lesssim \int_{|x_n|^2 \le \epsilon^2} \exp(-\frac{1}{2}(1+\lambda)|x_n|^2)\prod_n \frac{\mathrm{d}x}{\sqrt{2\pi}}$$

$$\lesssim \prod_n \frac{1}{\sqrt{1+\lambda_n}},$$

where λ_n are the eigenvalues of A. Now remains to estimate the eigenvalues.

Additional technicality: the above is not really true, because $I + A$ has zero eigenvalues (corresponding to the scaling and translation symmetries of Q).

Stochastic dynamics

Stochastic KdV with additive space-time white noise:

$$\begin{aligned} \mathrm{d}u + (\partial_x^3 u + u\partial_x u)\,\mathrm{d}t &= \mathrm{d}W, \\ u(x,0) &= u_0(x). \end{aligned}$$

Local solutions exist (Oh, 2009). For KdV (no noise term $\mathrm{d}W$), we have an invariant measure (white noise). For $\mathrm{d}u = \mathrm{d}W$, the variance grows linearly in time.

Work in progress (with Oh, Quastel): combine invariance and exact distribution of the noise to obtain global almost sure solutions.

Thank you for listening!