Stochastic Homogenization for Reaction-Diffusion Equations

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Joint Work with Andrej Zlatoš

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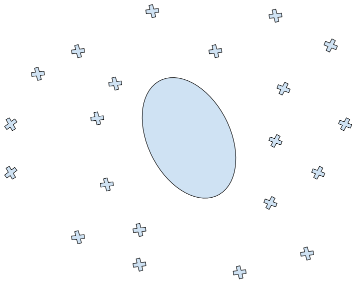
Motivation: Forest Fires



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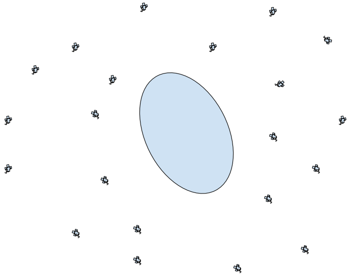

1. A PDE

Let $u(t,x)$ denote the temperature at $(t,x) \in (0,\infty) \times \mathbb{R}^d$.

$$\begin{cases} u_t - \Delta u = f(x,u) & \text{in} \quad (0,\infty) \times \mathbb{R}^d, \\ u(0,x) \approx \chi_{\Theta_0} & \text{on} \quad \mathbb{R}^d, \end{cases}$$

for $\Theta_0 \subseteq \mathbb{R}^d$ open and bounded. For each $x \in \mathbb{R}^d$, $f(x,\cdot)$:

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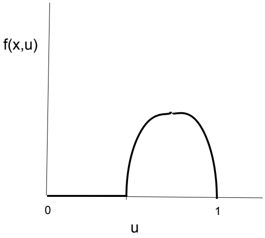
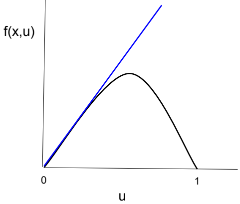


KPP



1. A PDE to Model Combustion

Let $u(t,x)$ denote the temperature at $(t,x) \in (0,\infty) \times \mathbb{R}^d$.

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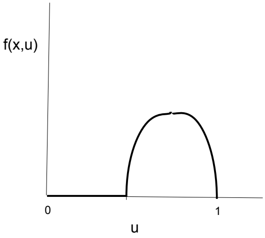

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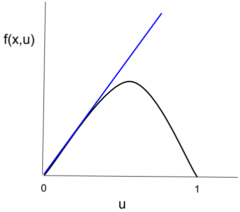

2. Conveying a Random Environment

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- $f(x, u, \omega)$ is an ignition reaction OR KPP reaction,
- $f_0(u) \leq f(x, u, \omega) \leq f_1(u)$, where $f_0, f_1 : [0, 1] \to \mathbb{R}$ are some fixed deterministic, homogeneous reactions of the same type as $f(x, u, \omega)$.

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Stationarity and Ergodicity (SE):

- $f(\cdot, u, \cdot)$ is stationary, i.e. there exists a measure-preserving group of transformations $\{\mathcal{T}_y\}_{y \in \mathbb{R}^d} : \Omega \to \Omega$ so that for all $u \in \mathbb{R}$,

$$f(x + y, u, \omega) = f(x, u, \mathcal{T}_y \omega).$$

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- $(\Omega, \mathcal{F}, \mathbb{P})$ is ergodic with respect to $\mathcal{T}_y$. In other words, if there exists an event $E \in \mathcal{F}$ so that

$$E = \mathcal{T}_y E \quad \text{for all} \quad y \in \mathbb{R}^d,$$

then $\mathbb{P}[E]$ is either 0 or 1.

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Q: What happens as $\varepsilon \to 0$?

Goal of Homogenization

Identify deterministic open sets $\{\Theta_t\}_{t>0}$ such that almost surely and locally uniformly away from the boundary $\Gamma_t := \partial\Theta_t$,

$$\lim_{\varepsilon\to 0} u^\varepsilon(t,x,\omega) = \begin{cases} 1 & \text{if} \quad x \in \Theta_t \\ 0 & \text{if} \quad x \in \mathbb{R}^d \setminus \overline{\Theta_t}. \end{cases}$$

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Use Viscosity Solutions interpretation.

Equivalent Goal:

Identify a deterministic function $c^* : \mathbb{S}^{d-1} \to (0,\infty)$ such that almost surely and locally uniformly in space-time (away from certain boundaries),

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where $\overline{u}$ is the unique viscosity solution of

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Barles, Soner, and Souganidis: $\overline{u}(t, x) = \chi_{\Theta_t}(x)$

Results

Theorem (Lions, Souganidis, '05)

Assume $(\Omega, \mathcal{F}, \mathbb{P})$ *is (SE),* $f(\cdot, \cdot, \omega)$ *is KPP. Then for* $\mathbb{P}$*-a.e.* ω*, homogenization holds.*

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- Stochastic homogenization for viscous HJ equations with convex Hamiltonians is well-understood.

Theorem (L., Zlatoš, '17)

Assume $(\Omega, \mathcal{F}, \mathbb{P})$ *is (SE),* $f(\cdot, \cdot, \omega)$ *is ignition,* $d \leq 3$*, and certain additional assumptions***. Then for* $\mathbb{P}$*-a.e.* ω*, homogenization holds.*

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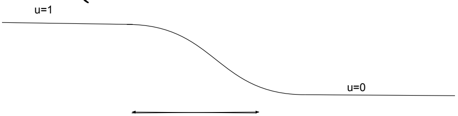

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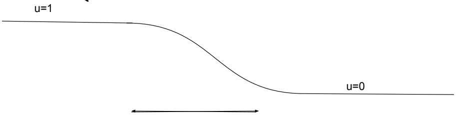

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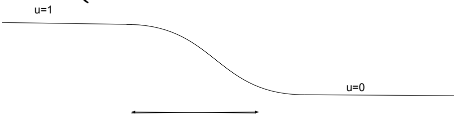

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$$x \to \frac{x}{\varepsilon} \approx xt \quad \Rightarrow \quad L_{u,\eta}(t) \sim o(t).$$

Theorem (Zlatoš, '14)

Let u solve

$$\begin{cases} u_t - \Delta u = f(x,u) & \textit{in} \quad (0,\infty)\times\mathbb{R}^d, \\ u_t \geq 0 & \textit{in} \quad (0,\infty)\times\mathbb{R}^d. \end{cases}$$

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For $d > 3$, this is not in general true! There exist reactions $f(\cdot,\cdot,\omega)$ with $\omega \in \Omega$ such that

$$L_{u,\eta,\omega}(t) \sim Ct$$

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   How will we relate c^* to the reaction-diffusion PDE?
2. Given c^*, show that $u^\varepsilon \to u$ in the appropriate sense.
   This is a completely deterministic PDE argument relying upon the theory of viscosity solutions and generalized front propagation.

Definition: Front Speeds

Fix $e \in \mathbb{S}^{d-1}$, and let $u(\cdot,\cdot,\omega)$ solve

$$\begin{cases} u_t - \Delta u = f(x,u,\omega) & \text{in} \quad (0,\infty)\times\mathbb{R}^d, \\ u(0,x,\omega) = \chi_{\{x\cdot e\leq 0\}}(x) & \text{on} \quad \mathbb{R}^d. \end{cases}$$

The front speed $c^*(e) > 0$ is the deterministic constant such that for $\mathbb{P}$-a.e. ω, for any $K \subseteq \mathbb{R}^d$ compact, for any $\delta > 0$,

$$\lim_{t\to\infty} \inf_{K\subseteq\{x\cdot e\leq c^*(e)-\delta\}} u(t,xt,\omega) = 1$$
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Observe: Initial data and front speeds are invariant with respect to hyperbolic scaling.

Key Difficulties

- Heterogeneous Setting. If the right hand side is $f(u)$, a traveling front with speed c satisfies

$$u(t,x) = U(x \cdot e - ct)$$

solves the PDE and

$$\lim_{s \to -\infty} U(s) = 1 \quad \lim_{s \to \infty} U(s) = 0.$$

If (U,c) is a traveling front pair, then c satisfies our definition of front speeds.

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If (U, c) is a traveling front pair, then c satisfies our definition of front speeds.

There is an analogous type of solution (pulsating front) for right hand side $f(x,u)$ when $f(\cdot, u)$ is periodic.

Key Difficulties

- Heterogeneous Setting. If the right hand side is $f(u)$, a traveling front with speed c satisfies

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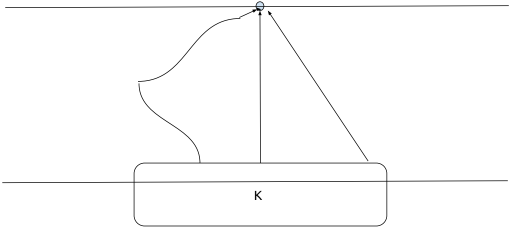
No such solutions exist for general heterogeneous right hand side $f(x,u)$.

- Front-Like Initial Data and Higher Dimensions:



Front speeds in random media in one dimension: Nolen and Ryzhik, Zlatoš

Definition: Spreading Speeds

Fix $e \in \mathbb{S}^{d-1}$, and let $u(\cdot,\cdot,\omega)$ solve

$$\begin{cases} u_t - \Delta u = f(x,u,\omega) & \text{in} \quad (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = \theta_0\chi_{B_R} & \text{on} \quad \mathbb{R}^d, \end{cases}$$

for R sufficiently large. Then we say $w(e)$ is the *spreading speed* in direction e if for $\mathbb{P}$-a.e. ω, for any $\delta > 0$,

$$\lim_{t\to\infty} u(t,(w(e)-\delta)te,\omega) = 1,$$
$$\lim_{t\to\infty} u(t,(w(e)+\delta)te,\omega) = 0.$$

First Passage Times for Reaction-Diffusion Equations

Define

$$\tau(0, y, \omega) := \inf\left\{t : u(t, x, \omega) \geq \theta_0 \chi_{B_R(y)}\right\}.$$

By the subadditive ergodic theorem, there exists a deterministic $\overline{\tau}(e)$ such that for $\mathbb{P}$-a.e. ω,

$$\lim_{n\to\infty} \frac{\tau(0, ne, \omega)}{n} = \overline{\tau}(e).$$

Then

$$w(e) := \frac{1}{\overline{\tau}(e)}$$

satisfies the definition of spreading speed.

All Directions at Once: The Wulff Shape

Proposition

Let $u(\cdot,\cdot,\omega)$ *solve*

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for R *sufficiently large. Define*

$$\mathcal{S} := \{se : 0 \le s \le w(e)\},$$

a convex set. For $\mathbb{P}$*-a.e.* ω*, for every* $\delta > 0$*, for* t *sufficiently large,*

$$(1-\delta)t\mathcal{S} \subseteq \left\{x : u(t,x,\omega) = \frac{1}{2}\right\} \subseteq (1+\delta)t\mathcal{S}.$$

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Question: How do we move from a speed for compactly-supported initial data to a speed for half-space initial data?

Recovery of Front Speeds

In the periodic setting, Freidlin-Gärtner formula says:

$$w(e) = \inf_{\substack{e' \in \mathbb{S}^{d-1}, \\ e' \cdot e > 0}} \frac{c^*(e')}{e' \cdot e}$$

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$$c^*(e) := \sup_{\substack{e' \in \mathbb{S}^{d-1}, \\ e' \cdot e > 0}} w(e') e' \cdot e$$

The additional assumptions* guarantee that the Wulff Shape $\mathcal{S}$ has no corners, so it has tangents in all directions. This is enough to show that $c^*(e)$ defined in this way is the front speed.

Example where Homogenization Holds: Isotropic Environment

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<u>Canonical Example:</u> Poisson Point Process

Let $\mathcal{P}(\omega) := \{x_n(\omega)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ denote a collection of points distributed by a Poisson point process with intensity 1. Then we have

$$f(x,u,\omega) \approx f_1(u)\chi_{B_1(\mathcal{P}(\omega))} + f_0(u)(1-\chi_{B_1(\mathcal{P}(\omega))})$$

Common Theme: Convexity

Let

$$\overline{H}(p) := c^* \left(\frac{p}{|p|} \right) |p|.$$

- For all solvable cases of stochastic homogenization for reaction-diffusion equations (solvable ignition and all KPP), $\overline{H}(p)$ is convex.

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- For general ignition, will likely need to strengthen some assumptions to obtain general homogenization.

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- Can we extend to time-dependent reactions? More general coefficients?
- Can we quantify the convergence in these statements? In particular, can we quantify the fluctuations to the front-like interface?

Thank you very much for your attention!