Geometry of polymers in the large deviation regime

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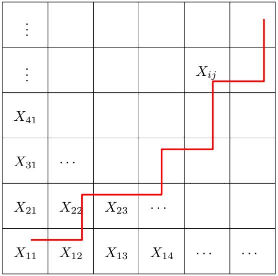

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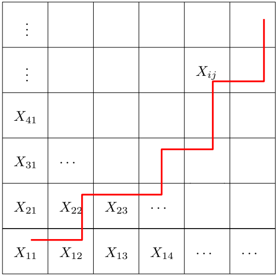

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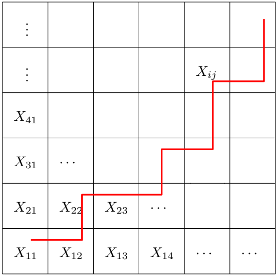

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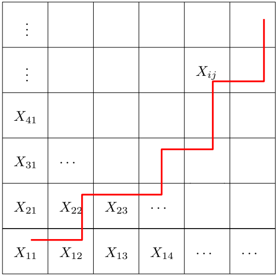

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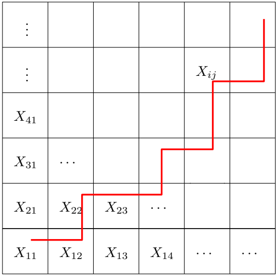

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Connections to particle systems and growth process

- Totally asymmetric exclusion process on $\mathbb{Z}$ (TASEP): particles at rate 1 jump to the right provided the site is empty.

Exponential LPP is equivalent to this model, where the passage time between $(0,0)$ and (N,N) denotes the time taken by the N^{th} particle to reach 0 starting from wedge initial conditions.

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- Under mild conditions, Poincaré inequality ensures that $\mathrm{Var}\, T_n = O(n)$.
- Γ_n w.h.p. has deviation $o(n)$ from the straight line joining $(0,0)$ to (n,n) under strict convexity of the limit shape boundary at $(1,1)$.

KPZ universality predictions

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- and much more...

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Based on bijections, exact formulae and connections to algebraic combinatorics, representation theory, determinantal processes, random matrix theory.

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Poissonian LPP

- RSK correspondence to Young Tableaux.
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- Transversal fluctuation exponent of $2/3$ is also rigorously known.

Large Deviations for polymer weights

Questions

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Upper tail large deviation: $T_n \geq (\mu + \delta)n$

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Lower tail large deviation: $T_n \leq (\mu - \delta)n$

- Large deviation speed is n^2 under minimal conditions.

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- For $\delta > 0$, $\lim_n \frac{\log \mathbb{P}(T_n \geq (2+\delta)n)}{n} = -I_u(\delta)$ where $I_u(\cdot)$ is an increasing convex function with $I(0) = 0$.

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- Via RSK, this is exactly the number of permutations with a given length for the longest increasing subsequence.

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- The passage time has the same law as the leading eigenvalue of the well known Wishart Ensemble.
- Coupling with TASEP also has been exploited in analyzing the upper tail. Seppäläinen (1998)

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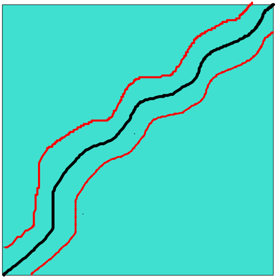
- Follows from the convexity of rate function. Can also be deduced for Exponential LPP.
- Corresponding question for lower tail was left open.

Result for Exponential LPP



- Let $\gamma : [0, 1] \to [0, 1]$ be a continuous increasing surjection.
- For $\varepsilon' > 0$, let

$$\gamma_n^{\varepsilon'} = \{(x, y) \in [0, n]^2 \cap \mathbb{Z}^2 : |y - n\gamma(n^{-1}x)| \leq \varepsilon' n\}.$$

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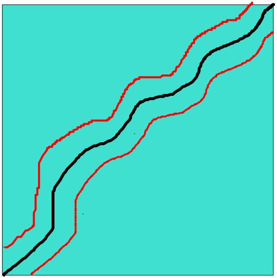

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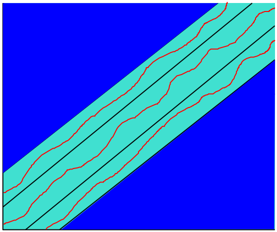
Theorem (Basu, G., Sly (2017))

Fix $\delta \in (0,4)$. Given any γ as above, and $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that for all large enough n,

$$\mathbb{P}(\Gamma_n \subseteq \gamma_n^{\varepsilon'} \mid T_n \leq (4-\delta)n) \leq \varepsilon.$$

n^2 speed for $\{T_n \leq (4-\delta)n\}$.



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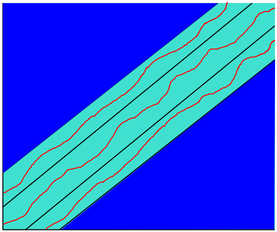

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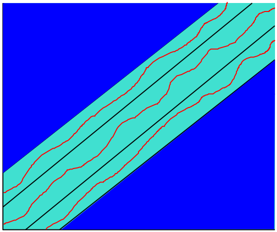

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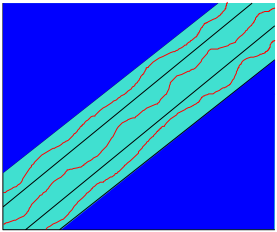

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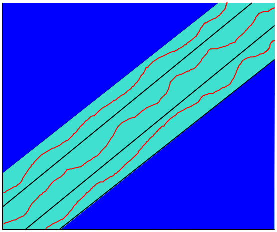

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The above argument already appeared in Kesten's work on First Passage Percolation.

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3. In turn this suggests that the polymer on the conditional environment should not be contained in a set of size $o(n^2)$ with high probability.
4. Formalizing this heuristic requires two ingredients.

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Proposition

Fix $\delta \in (0,4)$. Given any $\varepsilon > 0$ there exists $H > 0$ such that

$$\mathbb{P}\left(T_n \geq (4-\delta)n - \frac{H}{n} \mid T_n \leq (4-\delta)n\right) \geq 1 - \varepsilon.$$

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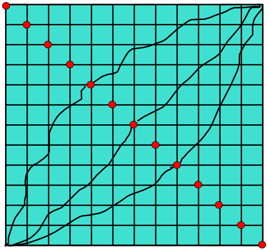

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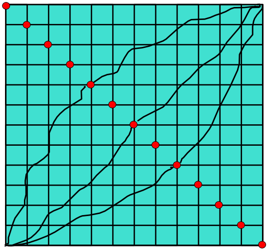

M_v is precisely the value that would make the longest path passing through v have weight $(4 - \delta)n$.

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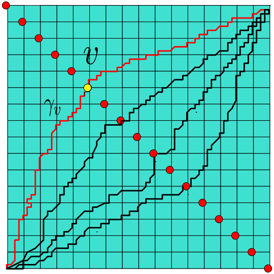

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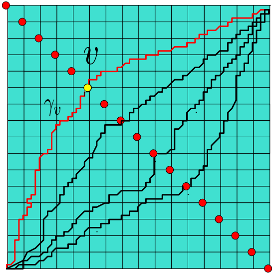

Now the theorem follows from an observation similar to the one which says maximum of n independent $U[0,1]$ variables is typically like $1 - O(\frac{1}{n})$.

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Proposition

Fix $\delta \in (0,4)$. Given any H and $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that for every deterministic set $A \subseteq [0,n]^2 \cap \mathbb{Z}^2$, with $|A| \le \varepsilon' n^2$ we have

$$\mathbb{P}\left(T_n(A) \ge (4-\delta)n - \frac{H}{n} \mid T_n \le (4-\delta)n\right) \le \varepsilon.$$

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Theorem (Basu, G., Sly (2017))

Fixing $\delta \in (0, 2)$, *for any increasing continuous* $\gamma : [0, 1] \to [0, 1]$ *with* $\gamma(0) = 0$ *and* $\gamma(1) = 1$, *there exists* $\varepsilon > 0$, *such that*

$$\mathbb{P}(\mathcal{E}_{\gamma,n} \mid T_n \leq (2 - \delta)n) \to 1$$

as $n \to \infty$, *where* $\mathcal{E}_{\gamma,n}$ *denotes the event that there exists a polymer* Γ_n *between* $(0, 0)$ *and* (n, n) *that is not contained in* γ_n^{ε}.

Beyond integrable settings

- Proofs do not use any inputs from integrable probability.
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Let F be a probability measure on $[0,\infty)$ that has continuous and non-increasing density and enough moments (or log-concave density). For $\delta \in (0,\mu_F)$ and $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that for all $\gamma : [0,1] \to [0,1]$ surjective and increasing one has

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- The key thing analyzed is the conditional distribution of the sum of a bunch of i.i.d. random variables conditioned on their projection on the unit L_1 ball.

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Thank You