The maxima of the Riemann zeta function on a short interval of the critical line

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joint work with
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Recent Trends in Continuous and Discrete Probability
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Outline

1. Maximum. The Riemann zeta function in a short interval
2. Free Energy. Moments of zeta and Freezing
3. Ideas of Proof.
4. Work in Progress and Open Questions.

Riemann zeta function

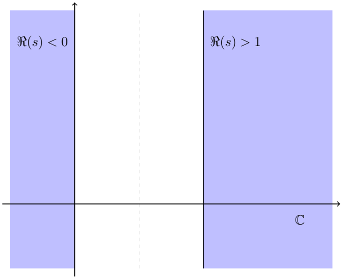
The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \mathrm{Re}(s) > 1$$
$$= \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$$

It can be analytically continued to a meromorphic function on $\mathbb{C}$

$$\zeta(s) = \chi(s)\zeta(1-s)$$

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)$$



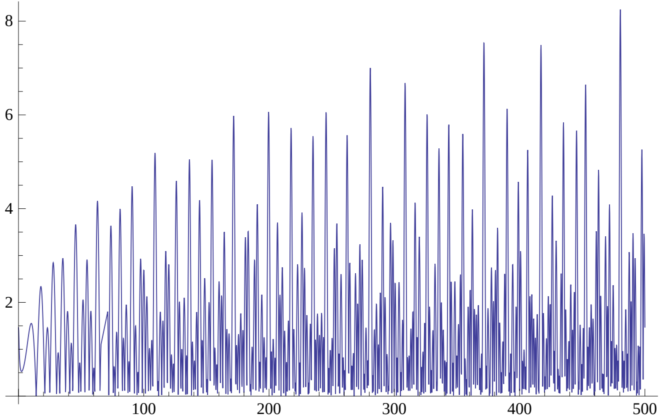


Figure: $|\zeta(1/2 + it)|$ for $t \in [0, 500]$

Zeta Function as a Random Field

Two helpful facts for the heuristics:

1. Selberg CLT If τ is $U([T, 2T])$

$$\log|\zeta(1/2 + i\tau)| \sim \text{Gaussian with variance } \frac{1}{2}\log\log T$$

2. Number of zeros in $[0, T]$:

$$\frac{T}{2\pi}\log\frac{T}{2\pi} + \mathrm{O}(\log T)$$

Naive Approximation (for high values)

Zeta on $\tau + [0, 1]$ can be seen as
$\log T$ Gaussians with variance $\frac{1}{2}\log\log T$.

Zeta Function as a Log.-Corr. Random Field

Multivariate Selberg CLT

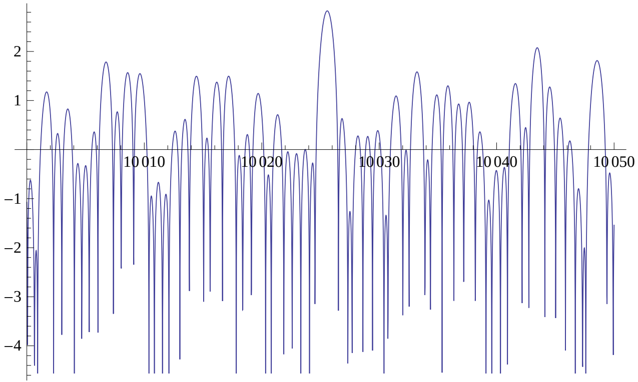
Theorem (Bourgade '09)

$$\frac{\log|\zeta(1/2+i(\tau+h))|}{\sqrt{\frac{1}{2}\log\log T}}, h \in [0,1]$$

converges in fin.-dim. distributions to a log-corr. Gaussian field

if $|h-h'| \sim (\log T)^{-\alpha}$, *covariance in the limit is* α, $0<\alpha<1$

1. Maximum in a typical short interval



Fyodorov-Hiary-Keating Conjecture

Conjecture (Fyodorov-Hiary-Keating '12)

If τ is sampled uniformly on $[T, 2T]$, then as $T \to \infty$

$$\max_{h \in [0,1]} \log |\zeta(1/2 + i(\tau + h))| = \log \log T - \frac{3}{4} \log \log \log T + \mathcal{M}_T$$

where the r.v.'s $(\mathcal{M}_T)$ converges in distribution.

- Should also hold for the imaginary part.
- Proved up to tightness in the case of log of characteristic polynomial of CUE: A-Belius-Bourgade '15, Paquette-Zeitouni '16, Chaibbi-Madaule-Najnudel '16.

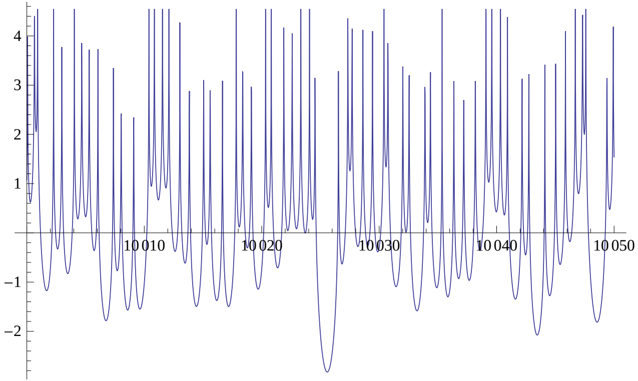
Fyodorov-Hiary-Keating Conjecture

Theorem (A-Belius-Bourgade-Radziwiłł-Soundararajan '17)
If τ is sampled uniformly on $[T, 2T]$,

$$\lim_{T\to\infty} \frac{1}{\log\log T} \cdot \max_{h\in[0,1]} \log|\zeta(1/2 + i(\tau + h))| = 1 \quad \textit{in probability}$$

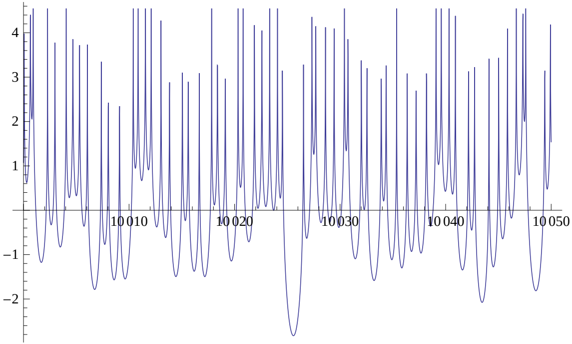
- In parallel, Najnudel proved the result (and the one for the imaginary part) conditionally on the Riemann Hypothesis.

2. Free Energy and Moments



Zeta function as a Disordered System

One can see of $-\log|\zeta(\tau + ih)|$, $h \in [0, 1]$, as an energy landscape of a disordered system.



FHK Conjecture (Free Energy)

Conjecture (FHK '12)

$$\lim_{T\to\infty} \frac{1}{\log\log T} \log \int_0^1 |\zeta(\frac{1}{2} + \mathrm{i}(\tau + \mathrm{h}))|^\beta \mathrm{dh} = \begin{cases} \frac{\beta^2}{4} & \text{if } \beta < 2 \\ \beta - 1 & \text{if } \beta \geq 2 \end{cases}$$

- Related to the entropy of high points: for $0 < \gamma < 1$,

$$\mathrm{Leb}\{h \in [0,1] : |\zeta(\frac{1}{2} + \mathrm{i}(\tau + \mathrm{h}))| > (\log \mathrm{T})^\gamma\} \approx (\log \mathrm{T})^{-\gamma^2} .$$

- A-Radziwiłł'18: Can be proved using similar methods unconditionally.
- Consistent with a freezing phase transition. (REM/Log-Correlated)

Moments of zeta

For a given point $\tau \sim U[T, 2T]$, it is expected that

$$\mathbb{E}\left[|\zeta(\frac{1}{2} + \mathrm{i}\tau)|^\beta\right] \sim C_\beta (\log T)^{\beta^2/4} \sim C_\beta \exp\left(\frac{\beta^2}{2} \cdot \frac{1}{2} \log\log T\right)$$

- Consistent with Selberg CLT.
- Known unconditionally for $\beta = 2$ and $\beta = 4$.
- Known on RH for all β. Large Deviation
- Lindelöf Hypothesis

$$\mathbb{E}\left[|\zeta(\frac{1}{2} + \mathrm{i}\tau)|^\beta\right] = O(T^\varepsilon) \quad \forall \varepsilon > 0$$

3. Sketch of proof for Upper Bound

$$\max_{h\in[0,1]} |\zeta(1/2 + \mathrm{i}(\tau + \mathrm{h})| < (\log \mathrm{T})^{1+\varepsilon}$$

$$\int_0^1 |\zeta(\frac{1}{2} + \mathrm{i}(\tau + \mathrm{h}))|^{\beta} \mathrm{dh} < (\log \mathrm{T})^{\mathrm{f}(\beta)+\varepsilon}$$

A basic Sobolev inequality

For $f \in H^1(\mathbb{R})$, the maximum is controlled by $L^2(\mathbb{R})$-norms

$$\|f\|_\infty^2 \leq \|f\|_2 \cdot \|f'\|_2 \ .$$

By applying this on $[0,1]$, then taking expectation:

$$\mathbb{E}\left[\max_{h\in[0,1]} |f(\tau+h)|^2\right] \ll \underbrace{\left(\mathbb{E}|f(\tau)|^2\right)^{1/2} \cdot \left(\mathbb{E}|f'(\tau)|^2\right)^{1/2}}_{\ll (\log T)^2} + \underbrace{\mathbb{E}|f(\tau)|^2}_{\ll \log T}$$

Take $f(t) = \zeta(\frac{1}{2} + \mathrm{i}t)$.

- The upper bound follows by Chebyshev's inequality.

3. Sketch of proof for Lower Bound

$$\max_{h\in[0,1]} |\zeta(1/2 + \mathrm{i}(\tau + \mathrm{h})| > (\log \mathrm{T})^{1-\varepsilon}$$

$$\int_0^1 |\zeta(\frac{1}{2} + \mathrm{i}(\tau + \mathrm{h}))|^{\beta} \mathrm{dh} > (\log \mathrm{T})^{\mathrm{f}(\beta)-\varepsilon}$$

Basic Principles

For $\sigma > 1$

$$\zeta(\sigma + \mathrm{i}t) = \prod_p \left(1 - p^{-\sigma - \mathrm{i}t}\right)^{-1} \quad .$$

For $0 < \sigma < 1$ this holds after truncation up to error

$$\log |\zeta(\sigma + \mathrm{i}t)| = \sum_{p \leq X} p^{-\sigma - \mathrm{i}t} + Error$$

1. Off-axis: Relate max on $\sigma = 1/2$ to $\sigma = 1/2 + \delta$.
2. Dirichlet polynomials: Relate ζ to sum for $X \approx T$
3. Log-corr. process: $(\sum_{p \leq X} p^{-\sigma - \mathrm{i}(\tau + h)}, h \in [0, 1])$ is log-correlated.

Maximum off-axis

Since the function is ∼analytic, the Poisson kernel on the line gives

$$\zeta(\sigma + \mathrm{i}t) = \int_{-\infty}^{\infty} \zeta\left(1/2 + \mathrm{i}u\right) \cdot \underbrace{\frac{1}{\pi} \frac{\sigma - 1/2}{(u-t)^2 + (\sigma - 1/2)^2}}_{\sim \textit{Cauchy} \;\; f_\sigma(u)} \mathrm{d}u \; .$$

This implies that if ζ is large at $\sigma + \mathrm{i}t$, then it must be large in an interval of the critical axis.

Relating ζ to truncated sum

Following Selberg's CLT by Radziwiłł & Soundararajan.

$$M(s) := \prod_p (1 - p^{-s}) \;\text{ for Re } s > 1,$$

Write $\widetilde{M}$ for M with a cutoff on the primes X:

$$\mathbb{E}\left|\widetilde{M}(\sigma + \mathrm{i}\tau)\zeta(\sigma + \mathrm{i}\tau) - 1\right|^2 = \mathrm{o}(1)$$

And using arithmetic argument:

$$\mathbb{E}\left|\widetilde{M}(\sigma + \mathrm{i}u) - \exp(\sum_{p \le X} p^{-\sigma - \mathrm{i}u})\right|^2 = \mathrm{o}(1)\ .$$

Using Sobolev-type argument as before, we can relate the maximum to the L^2 norms.

Dirichlet polynomial as a Gaussian

Let $S_T = \mathrm{Re} \sum_{p \le X} p^{-\sigma - \mathrm{i}\tau}$ and $\sigma = 1/2 + \delta$

$$\begin{aligned}
\mathbb{E}[S_T^2] &= \frac{1}{2} \sum_{p \le X} p^{-2\sigma} + \sum_{p \ne q} \mathrm{O}\left((pq)^{-1/2} \frac{1}{T} \int_T^{2T} e^{\mathrm{i}t(\log p - \log q)} \mathrm{d}t \right) \\
&= \frac{1}{2} \sum_{p \le X} p^{-2\sigma} + \mathrm{O}(X^2/T) \\
&= \frac{1}{2} \log\log X + \mathrm{O}(\delta \log X) + \mathrm{O}(X^2/T)
\end{aligned}$$

since density of primes is $(\log p)^{-1}$ (PNT).
Take $X = \exp((\log T)^{1-\varepsilon})$ and $\sigma = \frac{1}{2} + (\log T)^{\varepsilon} / \log T$.

$$\mathbb{E}[S_T^2] = \frac{1}{2} \log\log T + \mathrm{O}(\varepsilon \log\log T)$$

- Large moments: Gaussian with error X^{2k}/T.
- Large deviation regime: $k \approx \log\log T$

Dirichlet polynomials as log-corrrelated process

Let $S_T(h) = \mathrm{Re} \sum_{p \le X} p^{-\sigma - \mathrm{i}(\tau+h)}$.

$$\mathbb{E}[S_T(h) S_T(h')] = \frac{1}{2} \sum_{p \le X} \frac{\cos(|h-h'| \log p)}{p^{2\sigma}} + \mathrm{o}(1)$$

$$= \frac{1}{2} \sum_{\log p \le |h-h'|^{-1}} p^{-2\sigma} + \mathrm{o}(1) \ .$$

$$\boxed{\mathbb{E}[S_T(h) S_T(h')] = \frac{1}{2} \log |h-h'|^{-1} + \mathrm{o}(1) \ .}$$

- This means that the value $S_T(h)$ on $\log T$ points h is an approximate log-correlated field.

Dirichlet polynomials as log-corrrelated process

- Finally, we use Kistler's multiscale second moment method

$$P_j(h) = \mathrm{Re} \sum_{p \in J_j} p^{-\sigma - \mathrm{i}(\tau + h)}$$

$$J_j = [\exp((\log T)^{\frac{j}{K}}), \exp((\log T)^{\frac{j+1}{K}})], \qquad j = 1, \dots, K-2,$$

We show:

$$\mathbb{P}\left(\exists h \in [0,1] : P_j(h) > \frac{\lambda}{K} \log\log T \text{ for all } 1 \leq j \leq K-3 \right) \to 1$$

- This works the same way for the entropy of high points/free energy.

4. Work in Progress and Open Questions

Work in Progress and Open Questions

1. Open. Prove the leading order of the imaginary part unconditionally.
   The L^2 approximation does not work here.
2. Open. Prove the second order of the maximum

$$\frac{1}{\log\log\log T}\max_{h\in[0,1]}\{\log|\zeta(1/2+i(\tau+h))|-\log\log T\} = \frac{3}{4}+o(1)\,.$$

3. Open. Freezing holds up to what interval size ?

$$\lim_{T\to\infty}\frac{1}{\log\log T}\log\int_0^1|\zeta(\frac{1}{2}+\mathrm{i}(\tau+\mathrm{h}))|^\beta \mathrm{dh} = \begin{cases}\frac{\beta^2}{4} & \text{if } \beta<2\\ \beta-1 & \text{if } \beta\geq 2\end{cases}$$

Thank you !

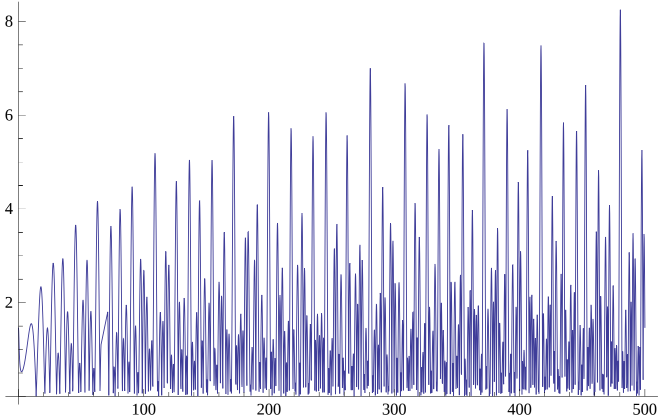


Figure: $|\zeta(1/2+it)|$ for $t \in [0, 500]$