Hypocoercive diffusion processes and gradient bounds

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- Probabilistic techniques (in particular coupling);
- Functional analytic techniques (Bakry-Émery calculus).

For both of those classical techniques, the ellipticity of the diffusion generator usually plays a major role. In the recent years, there have been extensive works to extend those approaches to hypoelliptic settings.

The Bakry-Émery criterion

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To address those questions and related hypercontractive estimates, Bakry-Émery introduced in 1985, a powerful calculus nowadays called the Γ-calculus.

The Bakry-Émery criterion

A central object to Bakry-Émery theory is the *carré du champ* operator

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$$\Gamma(f,g) = \frac{1}{2}\left(L(fg) - fLg - gLf\right)$$

and its iteration, the Bakry's operator

$$\Gamma_2(f,g) = \frac{1}{2}\left(L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\right).$$

The Bakry-Émery criterion

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The convergence is in L^2 (and in the entropic sense) with an exponential rate ρ.

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3. $W_2(P_t\mu, P_t\nu) \leq e^{-\rho t} W_2(\mu, \nu)$.

For instance, consider on $\mathbb{M}$ the Ornstein-Uhlenbeck type operator

$$Lf = \Delta f - \langle \nabla U, \nabla f \rangle$$

where $U : \mathbb{M} \to \mathbb{R}$.

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$$Lf = \Delta f - \langle \nabla U, \nabla f \rangle$$

where $U : \mathbb{M} \to \mathbb{R}$. The Bakry-Émery criterion for L is equivalent to

$$\mathbf{Ricci} + \nabla^2 U \geq \rho.$$

Hypoelliptic setting

The symmetry assumption on L is not important, but the Bakry-Émery criterion critically requires some form of ellipticity of the operator L.

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A toy model: the Kolmogorov diffusion

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where B_t is a Brownian motion in $\mathbb{R}^d$ with the variance σ^2. Its generator is

$$(Lf)(p, \xi) = \langle p, \nabla_\xi f(p, \xi) \rangle + \frac{\sigma^2}{2} \Delta_p f(p, \xi),$$

where Δ_p is the Laplace operator Δ on $\mathbb{R}^d$ acting on the variable p and ∇_ξ is the gradient on $\mathbb{R}^d$ acting on the variable ξ

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There is no lower bound on Γ_2. Due to the lack of ellipticity, Γ and Γ_2 are not in the same algebra !

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Let g be a Riemannian metric on $\mathbb{M}$. Consider then the second order differential bilinear form

$$\mathcal{T}_2(f) = \frac{1}{2}(L\|\nabla f\|_g^2 - 2\langle \nabla f, \nabla Lf \rangle_g), \quad f \in C_0^\infty(\mathbb{M}),$$

where ∇ denotes the Riemannian gradient for the metric g.

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3. *For every $\mu, \nu \in \mathcal{P}(\mathbb{M})$, and $t \geq 0$,*
$$W_2(P_t^* \mu, P_t^* \nu) \leq e^{-Kt} W_2(\mu, \nu).$$

Convergence in Wasserstein distance

Corollary

If there exists a positive constant K such that for every $f \in C_0^\infty(\mathbb{M})$,

$$\mathcal{T}_2(f) \geq K \|\nabla f\|_g^2, \tag{1}$$

then $P_t = e^{tL}$ has a unique invariant measure and converges exponentially fast to equilibrium in the W_2-Wasserstein distance.

Example: Linear diffusion with drift

Consider in $\mathbb{R}^n$ a stochastic differential equation

$$dX_t = b(X_t)dt + \sigma dB_t$$

where B is a Brownian motion in $\mathbb{R}^n$, σ a non necessarily invertible matrix and b a Lipschitz drift.

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Assume that there exists a constant positive definite matrix Σ and a constant $a > 0$ such that for every $x, y \in \mathbb{R}^n$,

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Then, there exists a unique $\mu \in \mathcal{P}(\mathbb{R}^n)$ such that for every $t \geq 0$, $P_t^*\mu = \mu$. Moreover, there exist constants $C_1, C_2 > 0$ such that for every $\nu \in \mathcal{P}(\mathbb{R}^n)$, and $t \geq 0$,

$$W_2(P_t^*\nu, \mu) \leq C_1 e^{-C_2 t} W_2(\mu, \nu),$$

Example: The kinetic Fokker-Planck equation

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. The kinetic Fokker-Planck equation with confinement potential V is the parabolic partial differential equation:

$$\frac{\partial h}{\partial t} = \Delta_v h - v \cdot \nabla_v h + \nabla_x V \cdot \nabla_v h - v \cdot \nabla_x h, \quad (x, v) \in \mathbb{R}^{2n}.$$

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It is the Kolmogorov-Fokker-Planck equation associated to the stochastic differential system

$$\begin{cases} dx_t = v_t dt \\ dv_t = -v_t dt - \nabla V(x_t) dt + dB_t, \end{cases}$$

where $(B_t)_{t \geq 0}$ is a Brownian motion in $\mathbb{R}^n$.

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where $(B_t)_{t \geq 0}$ is a Brownian motion in $\mathbb{R}^n$. Heuristically,

$$\frac{d^2 x_t}{dt} = -\frac{dx_t}{dt} - \nabla V(x_t) + \frac{dB_t}{dt},$$

describes the random motion of a particle in a force field with a white noise perturbation.

The kinetic Fokker-Planck equation

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$$L = \sum_{i=1}^{n} X_i^2 + Y,$$

where $X_i = \frac{\partial}{\partial v_i}$, $Y = -v \cdot \nabla_v + \nabla V \cdot \nabla_v - v \cdot \nabla_x$.

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where $X_i = \frac{\partial}{\partial v_i}$, $Y = -v \cdot \nabla_v + \nabla V \cdot \nabla_v - v \cdot \nabla_x$. The vectors

$$(X_1, \cdots, X_n, [Y, X_1], \cdots, [Y, X_n])$$

form a basis of $\mathbb{R}^{2n}$ at each point.

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form a basis of $\mathbb{R}^{2n}$ at each point. Hence L is hypoelliptic.

The kinetic Fokker-Planck equation

The operator L admits for invariant measure the measure

$$d\mu = e^{-V(x)-\frac{\|v\|^2}{2}}dxdv.$$

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L is not symmetric with respect to μ.

Villani and the kinetic Fokker-Planck equation

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Theorem (Villani, 2002)

Assume that $\|\nabla^2 V\| \leq c(1 + \|\nabla V\|)$ *and that* μ *satisfies the classical Poincaré inequality*

$$\int_{\mathbb{R}^{2n}} \|\nabla f\|^2 d\mu \geq \kappa \left[\int_{\mathbb{R}^{2n}} f^2 d\mu - \left(\int_{\mathbb{R}^{2n}} f d\mu \right)^2 \right].$$

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Then, there exist constants $C > 0$ *and* $\lambda > 0$ *such that for every* f*, with* $\int_{\mathbb{R}^{2n}} f d\mu = 0$,

$$\int_{\mathbb{R}^{2n}} (P_t f)^2 d\mu + \int_{\mathbb{R}^{2n}} \|\nabla P_t f\|^2 d\mu \leq C e^{-\lambda t} \left(\int_{\mathbb{R}^{2n}} f^2 d\mu + \int_{\mathbb{R}^{2n}} \|\nabla f\|^2 d\mu \right)$$

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Theorem (B. 2016)

Assume that there exist constants $m, M > 0$ *such that*

$$m \leq \nabla^2 V \leq M$$

and $\sqrt{M} - \sqrt{m} \leq 1$. *Then, there exist constants* $C_1, C_2 > 0$ *such that for every* $\mu, \nu \in \mathcal{P}(\mathbb{R}^{2n})$, *and* $t \geq 0$,

$$W_2(P_t^* \mu, P_t^* \nu) \leq C_1 e^{-C_2 t} W_2(\mu, \nu).$$

Convergence in $H^1(\mu)$

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Theorem (B. 2013)

Assume that L admits an invariant probability measure μ. If there exist a metric g and positive constants K_1, K_2 such that for every $f \in C_0^\infty(\mathbb{M})$,

$$\mathcal{T}_2(f) \geq K_1 \|\nabla f\|_g^2 - K_2 \Gamma(f), \qquad (2)$$

and if μ satisfies a Poincaré inequality

$$\int f^2 d\mu - \left(\int f d\mu \right)^2 \leq \frac{1}{\lambda} \int \|\nabla f\|_g^2 d\mu,$$

Convergence in $H^1(\mu)$

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$$\int f^2 d\mu - \left(\int f d\mu \right)^2 \leq \frac{1}{\lambda} \int \|\nabla f\|_g^2 d\mu,$$

then e^{tL} converges exponentially fast to equilibrium in $H^1(\mu)$. The rate of convergence may moreover be estimated explicitly in terms of K_1, K_2 and λ.

Revisiting the kinetic Fokker-Planck equation

Applying the weak form of the generalized Bakry-Émery criterion to the kinetic Fokker-Planck semigroup yields Villani's result, and moreover provides a quantitative rate of convergence to equilibrium.

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Those generalized Bakry-Émery criteria have later been applied to more general kinetic type semigroup and used to prove, for the first time, convergence to equilibrium of those semigroups (B.-Tardif, 2017).

Functional inequalities

As in the usual Bakry-Émery theory, beyond proving quantitative rate of convergence to equilibrium for diffusion semigroups, the generalized Γ-calculus may also be used to produce functional inequalities.

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As in the usual Bakry-Émery theory, beyond proving quantitative rate of convergence to equilibrium for diffusion semigroups, the generalized Γ-calculus may also be used to produce functional inequalities. Consider the Kolmogorov toy operator

$$(Lf)(p,\xi) = \langle p, \nabla_\xi f(p,\xi)\rangle + \frac{\sigma^2}{2}\Delta_p f(p,\xi),$$

Functional inequalities

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Theorem (B.-Gordina-Mariano, 2018)

Let $f \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ be a Lipschitz function, then one has

$$\|\nabla_p P_t f\|^2 \leqslant \sum_{i=1}^{d} P_t \left(\frac{\partial f}{\partial p_i} + t \frac{\partial f}{\partial \xi_i} \right)^2,$$

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Theorem (B.-Gordina-Mariano, 2018)

Let $f \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ be a Lipschitz function, then one has

$$\|\nabla_p P_t f\|^2 \leqslant \sum_{i=1}^{d} P_t \left(\frac{\partial f}{\partial p_i} + t \frac{\partial f}{\partial \xi_i} \right)^2 ,$$

and

$$\|\nabla_\xi P_t f\|^2 \leqslant P_t \|\nabla_\xi f\|^2 .$$

Functional inequalities

Theorem (B.-Gordina-Mariano, 2018)

Let $f \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ be a non-negative bounded function. One has for $t > 0$

$$\sum_{i=1}^{d} \left(\frac{\partial \ln P_t f}{\partial p_i} - \frac{1}{2} t \frac{\partial \ln P_t f}{\partial \xi_i} \right)^2 + \frac{1}{12} t^2 \left(\frac{\partial \ln P_t f}{\partial \xi_i} \right)^2$$

$$\leqslant \frac{2}{\sigma^2 t P_t f} (P_t(f \ln f) - P_t f \ln P_t f).$$