On the l^p norm of the discrete Hilbert transform*

Rodrigo Bañuelos†

Purdue University
Department of Mathematics
West Lafayette, IN. 47906

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*Join with Mateusz Kwaśnicki, Wrocław University of Technolgy

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The world is full of (mathematical) inequalities

Given an inequality many questions, often very difficult, arise

- Is the inequality sharp?
- Is equality ever attained?
- If attained, what further information does one get about the quantities (extremizers) that give equality?
- If a "quantity" gives strict inequality, how "far" is it from an extremizers?

Classical (continuous) Hilbert transform–conjugate function

$$Hf(x) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x-y)}{y}\,dy = \tilde{f}(x)$$

Known for more than 100 years: H is a Fourier multiplier.

$$\widehat{Hf}(\xi) = i sign(\xi)\widehat{f}(\xi)$$

This gives

$$\|Hf\|_2 = \|f\|_2$$

Equivalent formulation in the unit circle:

$$Hf(\theta) = p.v.\,\frac{1}{\pi}\int_0^{2\pi} cot\left(\frac{\theta-\varphi}{2}\right) f(\varphi)d\varphi = \tilde{f}(\theta)$$

Discrete Hilbert transform, D. Hilbert early 1900's. $\{a_n, n \in \mathbb{Z}\}$ a sequence

$$\mathcal{H}a_n = \frac{1}{\pi}\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{a_{n-m}}{m}$$

1. $H : L^1(\mathbb{R}) \not\to L^1(\mathbb{R})$ and $H : L^\infty(\mathbb{R}) \not\to L^\infty(\mathbb{R})$
2. $\|Hf\|_p \leqslant C_p \|f\|_p, \quad 1 < p < \infty$ (M. Riesz 1924)
3. $m\{x \in \mathbb{R} : |Hf(x)| > \lambda\} \leqslant \frac{C}{\lambda}\|f\|_1, \quad \lambda > 0,$ (A. Kolmogorov, 1926)

In his paper, M. Riesz: L^p bound for H implies l^p bound for $\mathcal{H}$. That is:

$$\|\mathcal{H}a_n\|_p \leqslant C_p' \|a_n\|_p, 1 < p < \infty, \quad \left(\|a_n\|_p = \left(\sum_{n\in\mathbb{Z}} |a_n|^p \right)^{1/p} \right)$$

In fact he showed more: $\|\mathcal{H}\|_p \leqslant C\|H\|_p$ and $\|H\|_p \leqslant \|\mathcal{H}\|_p$.

Here $\|H\|_p$ and $\|\mathcal{H}\|_p$ are the operator norms on $L^p(\mathbb{R})$ and $l^p(\mathbb{Z})$, respectively.

E. C. Titchmarsh (1926)

1. $\|\mathcal{H}a_n\|_p \leqslant C_p\|a_n\|_p, \quad 1 < p < \infty$

2. $\|\mathcal{H}\|_p = \|H\|_p, \;\; 1 < p < \infty$

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1. $\|\mathcal{H}a_n\|_p \leqslant C_p \|a_n\|_p, \quad 1 < p < \infty$

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3. Titchmarsh the following year (1927): The part of my paper that proves $\|\mathcal{H}\|_p \leqslant \|H\|_p$ is wrong. Only gives $\|\mathcal{H}\|_p \leqslant C'\|H\|_p$, as M. Riesz.

4. Equality of the norms has been a conjecture since.

M. Kwaśnicki & R.B. 2017. Conjecture is correct: $\|\mathcal{H}\|_p \leqslant \|H\|_p$.

Equality was know for even powers (L. Grafakos 1994, E.Laeng 2007): $p = 2^n$ or $p = \frac{2^k}{2^k-1}$, $k = 1, 2, \cdots$.

Calderón-Zygmund (1952)

$$Tf(x) = p.v. \int_{\mathbb{R}^d} K(y) f(x-y) dy,$$

K a Calderón-Zygmund (C-Z) kernel. $\|T\|_p \leqslant C_{p,d}\|f\|_p$, $1 < p < \infty$
C-Z considered discrete versions: $f : \mathbb{Z}^d \to \mathbb{R}$, $n \in \mathbb{Z}^d$

$$\mathfrak{T}(n) = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} K(m) f(n-m),$$

and showed (as Riesz and Titchmarsh): $\|T\|_p \approx \|\mathfrak{T}\|_p$.

Many discrete operators in harmonic analysis have been studied: Bourgain, Magyar, Pierce, Stein, Waigner, ...

Lillian Pierce, 2009 Ph.D. Thesis (322 pages) "Discrete analogues in harmonic analysis," Princeton University

Theorem (Stylianus Pichorides (1972))

$$\|Hf\|_p \leqslant cot\left(\frac{\pi}{2p^*}\right)\|f\|_p, \quad 1 < p < \infty,$$

where $p^* = \max(p, p/(p-1))$ *and in fact,*

$$\|H\|_p = cot\left(\frac{\pi}{2p^*}\right)$$

Theorem (M. Kwaśnicki & R.B. 2017)

$$\|\mathcal{H}a_n\|_p \leqslant cot\left(\frac{\pi}{2p^*}\right)\|a_n\|_p, \quad 1 < p < \infty,$$

This and Pichorides: $\|\mathcal{H}\|_p \leqslant \|H\|_p$, $1 < p < \infty$

Proof uses (I) Orthogonal martingales and (II) Doob h-processes

M_t and N_t martingales on filtration of d–dimensional BM.

$$N_t = N_0 + \int_0^t K_s \cdot d\,B_s, \quad M_t = M_0 + \int_0^t H_s \cdot d\,B_s$$

Assume:

- $|N_0| \leqslant |M_0|$ and $|K_s| \leqslant |H_s|$ a.s. for all s (N subordination to M): $N << M$
- $K_s \cdot H_s = 0$ a.s. for all s (N, M orthogonal): $N \perp M$

Then (Bañuelos–Wang (1995):

$$\|N\|_p \leqslant cot\left(\frac{\pi}{2p^*}\right)\|M\|_p, \;\; 1 < p < \infty$$

and the constant is best possible.

Note: Under subordination only (no orthogonality) one has Burkholder's inequalities with his famous constant $(p^* - 1)$.

$$p_w(x,y)=\frac{1}{\pi}\frac{y}{(|x-w|^2+y^2)} = \text{(Poisson kernel)} \quad x,w\in\mathbb{R}, y\in(0,\infty)$$

$B_t=(X_t,Y_t)$ B.M. in $\mathbb{R}^2_+=\{x,y):x\in\mathbb{R},y>0\}$, $\tau=\inf\{t>0:Y_t=0\}$

$$u_f(x,y))=\frac{1}{\pi}\int_{\mathbb{R}} p_w(x,y)f(w)dw=\mathbb{E}_{(x,y)}\left(f(B_\tau)\right)=\mathbb{E}_{(x,y)}\left(f(X_\tau)\right)$$

Trivial observations

- (i) $u_f(x,y)$ harmonic in $\mathbb{R}^2_+$, (ii) $u_f(x,y)\to f(x),\ \ y\downarrow 0$, (iii) $u_f(x,y)\to 0,\ \ y\to\infty$
- $\pi y\,\mathbb{E}_{(x,y)}|f(B_\tau)|^p\to\int_{\mathbb{R}}|f(x)|^p dx$, as $y\to\infty$.
- v_f is the conjugate harmonic of u_f. Cauchy-Riemann

$$|\nabla u_f(x,y)|=|\nabla v_f(x,y)|.$$

$$\nabla v_f=\mathbb{H}\nabla u_f,\ \ \mathbb{H}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},$$

$$M_t^f = u_f(B_{t\wedge\tau}) = u_f(B_0) + \int_0^{t\wedge\tau} \nabla u_f(B_s)\cdot dB_s$$

$$(\mathbb{H}\star M^f)_t = v_f(B_{t\wedge\tau}) = v_f(B_0) + \int_0^{t\wedge\tau} \mathbb{H}\nabla u_f(B_s)\cdot dB_s$$

$(\mathbb{H}\star M^f)_t << M_t^f$, $(\mathbb{H}\star M)_t \perp M_t^f \Rightarrow$

$$\mathbb{E}_{(x,y)}|(\mathbb{H}\star M^f)_\tau|^p \leqslant \left(cot\left(\frac{\pi}{2p^*}\right)\right)^p \mathbb{E}_{(x,y)}|M_\tau^f|^p, \quad 1<p<\infty.$$

Same as

$$\mathbb{E}_{(x,y)}|Hf(B_\tau)|^p \leqslant \left(cot\left(\frac{\pi}{2p^*}\right)\right)^p \mathbb{E}_{(x,y)}|f(B_\tau)|^p, \quad 1<p<\infty.$$

Fix x, multiply both sides by πy and send y to infinity to get Pichorides inequality.

For general (constant or even variable coefficient) 2×2 matrix A.

Fix $(0, y) \in \mathbb{R}^2_+$. Define:

$$T^y_A f(x) = E_{(0,y)}\left(\int_0^\tau A\nabla u_f(B_s)\cdot dB_s | B_\tau = x\right)$$

$$Any\ A: \quad \left(E_{(0,y)}|T^y_A f(B_\tau|\right)^{1/p} \leqslant (p^*-1)\|A\| \left(E_{(0,y)}|f(B_\tau)|^p\right)^{1/p}$$

$$\|A\| = \|\sup_{|v|\leqslant 1}(|A(z,w)v|)\|_{L^\infty(\mathbb{R}^2\times[0,\infty))} < \infty,$$

(From Burkholder's inequality)

$$Orthogonal\ \ A: \quad \left(E_{(0,y)}|T^y_A f(B_\tau|\right)^{1/p} \leqslant cot\left(\frac{\pi}{2p^*}\right)\|A\| \left(E_{(0,y)}|f(B_\tau)|^p\right)^{1/p}$$

(From R.B. and Wang inequality)

As $y \to \infty$, we get (for any any $d \geqslant 1$)

$$T_A f(x) = \int_{\mathbb{R}^d} K(x,\tilde{x}) f(\tilde{x}) d\tilde{x},$$

$$K_A(x,\tilde{x}) = \int_0^\infty \int_{\mathbb{R}^d} 2w A(z,w) \nabla p_{\tilde{x}}(z,w) \cdot \nabla p_x(z,w) dz\, dw,$$

Simpler formula in the unit disc D, boundary $\partial D = \mathbb{T}$. BM starting at the origin $\tau = \inf\{t > 0 : B_t \in \mathbb{T}\}$. $f : \mathbb{T} \to \mathbb{R}$, u_f harmonic extension of f in D.

$$\begin{aligned} T_A f(e^{i\theta}) &= \mathbb{E}_0 \left(\int_0^\tau A \nabla u_f(B_s) \cdot dB_s | B_\tau = e^{i\theta} \right) \\ &= \frac{1}{\pi} \int_D \log \left(\frac{1}{|z|} \right) A \nabla u_f(z) \cdot \nabla h_\theta(z) dz \end{aligned}$$

where $h_\theta(z)$ is the Poisson kernel for the disc.

$$h_\theta(z) = \frac{(1 - |z|^2)}{|e^{i\theta} - z|^2}$$

$L = \{2\pi n : n \in \mathbb{Z}\}$

$$p_n(x,y) = \frac{1}{\pi} \frac{y}{(x - 2\pi n)^2 + y^2},$$

for $x \in \mathbb{R}$ and $y > 0$. We define

$$h(x,y) = \sum_{n \in \mathbb{Z}} p_n(x,y) = \frac{1}{2\pi} \frac{\sinh y}{\cosh y - \cos x}.$$

h positive harmonic. For $f : L \to \mathbb{R}$ defined by $f(2\pi n) = a_n$ (compact support)

$$u_f(x,y) = \sum_{n \in \mathbb{Z}} f(2\pi n) \frac{p_n(x,y)}{h(x,y)}, \quad x \in \mathbb{R}, \;\; y > 0$$

u_f is h-harmonic in $\mathbb{R}^2_+$:

$$\frac{1}{2} \Delta u_f(x,y) + \frac{\nabla h(x,y) \cdot \nabla u_f(x,y)}{h(x,y)} = 0.$$

$u_f(2\pi n, 0) = a_n$; u_f is the h-harmonic extension of $\{a_n\}$.

$Z_t = (X_t, Y_t) \in \mathbb{R}^2_+$, be the Doob's h B.M. Only exists $\mathbb{R}^2_+$ on $L \times \{0\}$.

$$\tau = \inf\{t \geqslant 0 : Y_t = 0\}.$$

$$dZ_t = dB_t + \frac{\nabla h(Z_t)}{h(Z_t)} dt.$$

Consider the martingale: $M_t^f = u_f(Z_{t\wedge\tau})$. Itô's formula gives

$$\begin{aligned} M_t^f &= M_0^f + \int_0^{t\wedge\tau} \nabla u_f(Z_s) \cdot dZ_s + \frac{1}{2}\int_0^{t\wedge\tau} \Delta u_f(Z_s) ds \\ &= M_0^f + \int_0^{t\wedge\tau} \nabla u_f(Z_s) \cdot dZ_s - \int_0^{t\wedge\tau} \frac{\nabla h(Z_s) \cdot \nabla u_f(Z_s)}{h(Z_s)} ds \\ &= M_0^f + \int_0^{t\wedge\tau} \nabla u_f(Z_s) \cdot dB_s. \end{aligned}$$

Any 2×2 matrix A the martingale transform of M_t is

$$\begin{aligned} (A \star M^f)_t &= \int_0^{t\wedge\tau} A\nabla u(Z_s) \cdot dB_s \\ &= \left(\int_0^{t\wedge\tau} A\nabla u(Z_s) \cdot dZ_s - \int_0^{t\wedge\tau} \frac{A\nabla u(Z_s) \cdot \nabla h(Z_s)}{h(Z_s)}\, ds\right) \end{aligned}$$

For $y > 0$, we now define an operator $\mathcal{T}^y$:

$$\mathcal{T}_A^y(f)(n) = \mathbb{E}_{(0,y)}[(A \star M^f)_\tau | X_\tau = 2\pi n]$$

1. For any 2×2 matrix A and $p \in (1, \infty)$,

$$\mathbb{E}_{(0,y)}|\mathcal{T}_A^y f(X_\tau)|^p \leqslant \mathbb{E}_{(0,y)}|(A \star M^f)_\tau|^p \leqslant (p^* - 1)^p \|A\|^p \mathbb{E}_{(0,y)}|f(X_\tau)|^p.$$

2. If A is orthogonal ($A\vec{v} \cdot \vec{v} = 0$, all $\vec{v} \in \mathbb{R}^2$)

$$\mathbb{E}_{(0,y)}|\mathcal{T}_A^y f(X_\tau)|^p \leqslant \mathbb{E}_{(0,y)}|(A \star M^f)_\tau|^p \leqslant \left(cot\left(\frac{\pi}{2p^*}\right)\right)^p \|A\|^p \mathbb{E}_{(0,y)}|f(X_\tau)|^p.$$

Proposition: As $y \to \infty$, $\mathcal{T}_A^y(f) \to \mathcal{J}_A(f)$ and

1. For any A,

$$\|\mathcal{J}_A(f)\|_{\ell^p(\mathbb{Z})} \leqslant (p^* - 1)\|A\| \|f\|_{\ell^p(\mathbb{Z})}$$

2. for any orthogonal A

$$\|\mathcal{J}_A(f)\|_{\ell^p(\mathbb{Z})} \leqslant cot(\frac{\pi}{2p^*})\|A\| \|f\|_{\ell^p(\mathbb{Z})}.$$

Theorem (M. Kwaśnicki & R.B.)

With $\mathbb{H} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *(orthogonal of norm 1)*

$$\mathcal{J}_{\mathbb{H}}(f)(n) = \sum_{m \in \mathbb{Z}} f(n-m)\mathcal{J}(n)$$

where

$$\begin{aligned}
\mathcal{J}(n) &= \int_{\mathbb{R}} \int_0^\infty \frac{2y}{h(x,y)} \mathbb{H}\nabla p_n(x,y) \cdot \nabla p_0(x,y) dy dx \\
&+ \int_{\mathbb{R}} \int_0^\infty 4y p_0(x,y) \mathbb{H}\nabla p_n(x,y) \cdot \nabla(\frac{1}{h(x,y)}) dy dx \\
&= \frac{1}{\pi n} \left(1 + \int_0^\infty \frac{2y^3}{(y^2 + \pi^2 n^2) \sinh^2 y} \, dy \right)
\end{aligned}$$

for $n \neq 0$*, and* $\mathcal{J}_0 = 0$*.*

Lemma (A fundamental fact)

The discrete Hilbert transform is the convolution of $\mathcal{J}$ with a probability kernel.

That is, there exists a nonnegative sequence $\{\mathcal{K}(n); n \in \mathbb{Z}\}$ with total mass 1 such that

$$\mathcal{H}f(n) = \sum_{m \in \mathbb{Z}} \mathcal{K}(n-m)\mathcal{J}_{\mathbb{H}}(f)(m),$$

Corollary

$$\|\mathcal{H}f(n)\|_{\ell^p(\mathbb{Z})} \leqslant \|\mathcal{J}_{\mathbb{H}}(f)\|_{\ell^p(\mathbb{Z})} \leqslant cot\left(\frac{\pi}{2p^*}\right)\|f\|_{\ell^p(\mathbb{Z})}, \quad 1 < p < \infty,$$

$p^* = \max(p, p/(p-1))$

Theorem (Burgess Davis 1974: Sharp Kolmogorov's inequality)

$$m\{x \in \mathbb{R} : |Hf(x)| > \lambda\} \leqslant \frac{D}{\lambda}\|f\|_1, \quad \forall \lambda > 0$$

$$D = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots} = \frac{\pi^2}{8\beta(2)} \approx 1.328434313301$$

($\beta(2)$ is Catalan's constant.) D is best possible.

For discrete $\mathcal{H}$ one has

$$\#\{n \in \mathbb{Z} : |\mathcal{H}a_n| > \lambda\} \leqslant \frac{C}{\lambda}\sum_{n\in\mathbb{Z}} |a_n|,$$

for all $\lambda > 0$, some $C > 0$.

Kwaśnicki & R.B. 2017

Conjecture: Best C is D. (We showed that $D \leqslant C$.)

$$\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{f(x-\varepsilon m)}{m}=Hf(x)$$

Fatou's lemma,

$$m\{x\in\mathbb{R}:|Hf(x)|>\lambda\}\leqslant\liminf_{\varepsilon\to 0^+} m\Big\{x\in\mathbb{R}:\Big|\frac{1}{\pi}\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{f(x-\varepsilon m)}{m}\Big|>\lambda\Big\}.$$

We write $x=\varepsilon y=\varepsilon(z+n)$ for $z\in[0,1)$ and $n\in\mathbb{Z}$. By Fubini,

$$\begin{aligned}
& m\Big\{x\in\mathbb{R}:\Big|\frac{1}{\pi}\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{f(x-\varepsilon m)}{m}\Big|>\lambda\Big\}\\
= \quad & \varepsilon\, m\Big\{y\in\mathbb{R}:\Big|\frac{1}{\pi}\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{f(\varepsilon(y-m))}{m}\Big|>\lambda\Big\}\\
= \quad & \varepsilon\int_0^1\#\Big\{n\in\mathbb{Z}:\Big|\frac{1}{\pi}\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{f(\varepsilon(z+n-m))}{m}\Big|>\lambda\Big\}dx.
\end{aligned}$$

We now apply weak-type to the sequence $a_n = f(\varepsilon(z+n))$ with z fixed. It follows that

$$m\left\{x \in \mathbb{R} : \left|\frac{1}{\pi} \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{f(x - \varepsilon m)}{m}\right| > \lambda\right\}$$

$$\leqslant \quad \frac{C\varepsilon}{\lambda} \int_0^1 \sum_{n \in \mathbb{Z}} |f(\varepsilon(z+n))| dz = \frac{C\varepsilon}{\lambda} \int_{-\infty}^{\infty} |f(\varepsilon y)| dy = \frac{C}{\lambda} \int_{-\infty}^{\infty} |f(x)| dx.$$

There is an absolutely summable sequence $(\mathcal{E}_n)$ such that

$$\mathcal{J}_{\mathbb{H}} = \mathcal{H} + \mathcal{H}\mathcal{E},$$

($\mathcal{H}\mathcal{E}$ convolution operator.) Further, $\mathcal{E}_n < 0$ for all $n \neq 0$, $\mathcal{E}_0 > 0$ and the sum of all $\mathcal{E}_n$ is zero. Write $\alpha = (1 + \mathcal{E}_0)^{-1}$, $\mathcal{G}_n = -\alpha\mathcal{E}_n$ for $n \neq 0$ and $\mathcal{G}_0 = 0$. Then $\mathcal{G}_n \geqslant 0$ for all n, the sum of all $\mathcal{G}_n$ is equal to $\alpha\mathcal{E}_0 = 1 - \alpha$, and

$$\mathcal{J}_{\mathbb{H}} = \mathcal{H} + (\mathcal{E}_0\mathcal{H} - \alpha^{-1}\mathcal{H}\mathcal{G}) = \alpha^{-1}\mathcal{H}(\mathcal{I} - \mathcal{G}).$$

Set

$$\mathcal{K} = \alpha \sum_{k=0}^{\infty} \mathcal{G}^k.$$

Furthermore, $\mathcal{K}$ is a convolution operator with kernel $(\mathcal{K}_n)$ such that $\mathcal{K}_n \geqslant 0$ for all n, and the sum of all $\mathcal{K}_n$ is equal to $\alpha \sum_{k=0}^{\infty}(1-\alpha)^k = 1$. Finally,

$$\mathcal{J}\mathcal{K} = \alpha^{-1}\mathcal{H}(\mathcal{I} - \mathcal{G}) \sum_{k=0}^{\infty} \alpha\mathcal{G}^k = \mathcal{H} \sum_{k=0}^{\infty} (\mathcal{I} - \mathcal{G})\mathcal{G}^k = \mathcal{H}.$$

Thank You!